A MINIMAL DEGREE LESS THAN 0′

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Clifford Spector in [4] proved that there exists a minimal degree less than 0″. J. R. Shoenfield in [3] asked: "Does there exist a minimal degree a such that $a \leqq 0'$?" We show that the answer to his question is yes! Our notation is that of [4].

We say that b strictly extends a if b and a are distinct sequence numbers, and if the sequence represented by b extends the one represented by a; we express this symbolically as SExt (b, a). If $\{a_0, a_1, a_2, \cdots\}$ is a sequence of sequence numbers such that for each i, a_{i+1} strictly extends a_i, then there is a unique function $f(n)$ such that for each i there is an m with the property that $\bar{f}(m) = a_i$; if $\{a_0, a_1, a_2, \cdots\} \subseteq S$, then we say $f(n)$ is a function associated with S. Spector in [4] obtained a function of minimal degree as the unique function associated with every member of a contracting sequence of sets of sequence numbers. Our construction is inspired by his, but it differs markedly from his in one respect: each one of our sets of sequence numbers will be recursively enumerable, whereas each one of his was recursive.

For each natural number c, let c^* be the unique, recursively enumerable set which has c as a Gödel number. There exists a recursive function $g(n)$ such that for each c, $g(c)$ is the Gödel number of the representing function of a recursive predicate $R_c(m, x)$ with the property that $x \in c^*$ if and only if $(Em)R_c(m, x)$. We define a recursive predicate $H(c, t, e, x, m, b, d)$ which is basic to our construction:

$$H(c, t, e, x, m, b, d) \equiv (i)_{i<2}(\text{SExt}((x)_i, t) \;\&\; R_c((m)_i, (x)_i) \;\&\; T_1^1((x)_i, e, b, (d)_i)) \;\&\; U((d)_0) \neq U((d)_1).$$

We define a partial recursive function $Y(c, t, e)$:

$$Y(c, t, e) = \begin{cases} \mu x H(c, t, e, (x)_0, (x)_1, (x)_2, (x)_3) & \text{if } (Ex)H(c, t, e, (x)_0, (x)_1, (x)_2, (x)_3) \\ \text{undefined otherwise.} \end{cases}$$

We define a recursively enumerable set of sequence numbers denoted by $W(c, t, e)$: (a) $t \in W(c, t, e)$ if t is a sequence number; (b) if $u \in W(c, t, e)$ and if $Y(c, u, e)$ is defined, then $(Y(c, u, e))_{0,0} \in W(c, t, e)$ and $(Y(c, u, e))_{0,1} \in W(c, t, e)$; and (c) every member of $W(c, t, e)$ is

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