# VECTOR FIELDS ON SPHERES

BY HIROSI TODA

Communicated by Deane Montgomery, March 27, 1961

1. The problem is to determine the maximal number of the independent continuous fields of tangent vectors on the unit $n$-sphere $S^n$. The number will be denoted by $\lambda(n)$.

$\lambda(n)$ is the maximal number of $k$ such that the boundary homomorphism $\Delta_{n,k}: \pi_n(S^n) \to \pi_{n-1}(O_{n,k})$ associated with the fibering $O_{n+1,k+1}/O_{n,k} = S^n$ is trivial, where $O_{n,k}$ denotes the Stiefel manifold of the orthogonal $k$-vectors ($k$-frames) in the real $n$-space $R^n$.

The fundamental conjecture for our problem is stated as follows.

CONJECTURE. Does $\lambda(n) = \lambda^*(n)$ for all $n > 0$?

Here, the conjectured values $\lambda^*(n)$ are defined as follows:

$$\lambda^*(n) = \lambda_r, \qquad \text{if } n \equiv 2^r - 1 \pmod{2^{r+1}},$$
$$\lambda_0 = 0, \qquad \lambda_1 = 1, \qquad \lambda_2 = 3, \qquad \lambda_3 = 7$$

and

$$\lambda_{r+4} = \lambda_r + 8.$$

It was known that the conjecture is true for the cases $r = 0, 1, 2, 3$ [4].

The obtained results on $\lambda(n)$ are the following.

THEOREM 1. (a) $\lambda^*(n) \leqq \lambda(n)$. (b) *If* $k = \lambda^*(n)$, *then the image of* $\Delta_{n,k}: \pi_n(S^n) \to \pi_{n-1}(O_{n,k})$ *coincides with the image of the composition* $i_* \circ J: \pi_k(SO(n-k-1)) \to \pi_{n-1}(S^{n-k-1}) \to \pi_{n-1}(O_{n,k})$ *of G. Whitehead's homomorphism* $J$ *and the homomorphism* $i_*$ *induced by the usual injection* $i: S^{n-k-1} \subset O_{n,k}$.

The first part (a) is provided by the recent work of Bott and Shapiro, *Clifford modules and vector fields on spheres* (mimeographed note), which states the existence of a continuous field of linear $\lambda^*(n)$-frames on $S^n$.

THEOREM 2. $\lambda^*(n) = \lambda(n)$ *if* $n \equiv 2^r - 1 \pmod{2^{r+1}}$ *for an integer* $r < 11$.

Then our problem is still open in question on the sphere $S^{2047}$.

THEOREM 3. $\lambda(2^i m - 1) \geqq \lambda(m-1) + 2^{i-1}$ *for* $i = 1, 2, 3, 4$.

COROLLARY. *If the above conjecture is not true for an* $n \equiv 2^r - 1 \pmod{2^{r+1}}$ *and* $r = 4s - 1$ ($s$: *positive integer*), *then the conjecture is not true for all* $n$ *of* $r \geqq 4s - 1$.