

ON THE EULER CHARACTERISTIC OF COMPACT COMPLETE LOCALLY AFFINE SPACES. II

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Communicated by I. M. Singer, March 11, 1961

The main result of this paper may be stated very simply: *A compact complete locally affine space has Euler characteristic zero.* In [1] we showed that if the radical of the fundamental group is nontrivial then the Euler characteristic is zero. Hence all that remains is to show that the radical of the fundamental group is indeed nontrivial. To do this one may as well limit oneself to the study of those compact completely locally affine spaces where the holonomy group is discrete and isomorphic to the fundamental group. For it is known that in all other cases the radical is nonzero (see [3]).

Let $A(n) = \text{GL}(n) \cdot R^n$ be the group of affine transformations of the n -dimensional affine plane V^n , where the dot denotes the semi-direct product and R^n is the n -dimensional vector space. Let Γ be a discrete subgroup of $A(n)$ such that V^n/Γ is a compact manifold. From [2], it is trivial that every compact complete locally affine space can be so realized. Hence we have the holonomy group of Γ , $h(\Gamma)$, isomorphic to $\Gamma R^n/R^n \subset \text{GL}(n)$, is discrete and isomorphic to Γ . Now $\Gamma \backslash A(n)$ can be identified with the principal bundle of all frames over M . Further, $\Gamma \backslash A(n)$ is sheeted by the images of the cosets of V^n in $A(n)$. This sheeting, since $h(\Gamma)$ is discrete, determines a fiber bundle over $h(\Gamma) \backslash \text{GL}(n)$ with fiber homeomorphic to V^n . We will call this fiber bundle B . Since the fiber is solid there exists a cross-section $\psi: h(\Gamma) \backslash \text{GL}(n) \rightarrow B$. Further each fiber inherits the structure of an affine plane. In addition this affine structure is preserved by the mappings of the fiber onto itself induced by the group of the bundle. Since the cross-section ψ exists, it is easy to verify that the group of the bundle is $h(\Gamma)$ acting on the fiber.

It is possible using the holonomy covering space in [2] to define $h(\Gamma)$ acting on the bundle B . Further for $h(\gamma) \in h(\Gamma)$ $h(\gamma): B \rightarrow B$ is a bundle map which acts trivially on the base space. Hence $h \cdot \psi$ is another section over $h(\Gamma) \backslash \text{GL}(n)$. Therefore the vector determined by $h \cdot \psi$ is invariant under $h(\Gamma)$ acting on the fiber. Hence, this shows that the images of the origin under Γ acting on V^n are left point-wise fixed by $h(\Gamma)$ acting on V^n . Since the images of the origin under Γ must span V^n for V^n/Γ to be compact, it follows that B is a trivial bundle. Using this fact one can show that Γ must be abelian.

Complete details will be presented elsewhere.