APPROXIMATION IN UNIFORM NORM BY SOLUTIONS OF ELLIPTIC DIFFERENTIAL EQUATIONS

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Introduction. Let G be an open subset of the Euclidean n-space E^n, G_1 an open subset with compact closure in G. If $n=2$ and G is the whole of E^2, an important circle of theorems in the theory of analytic functions associated with the names of Walsh, Hartogs-Rosenthal, Lavrentiev, Keldych, and Mergelyan deals with the possibility of approximating analytic functions on G_1 continuous on its closure, uniformly on G_1 by polynomials in the complex variable z. Mergelyan's theorem [1], the most general of these results, asserts that if $\overline{G}_1$ does not disconnect E^2, then every such analytic function is uniformly approximable by polynomials on $\overline{G}_1$. More generally, if we replace $\overline{G}_1$ by any compact subset K of E^2, Mergelyan's result asserts that if K does not disconnect E^2, then every continuous function on K which is analytic at every interior point of K is uniformly approximable on K by polynomials in z. In view of the classical theorem of Runge on uniform approximation of analytic functions on compact subsets of G_1 by polynomials, Mergelyan's theorem is equivalent to the assertion that each function $f(z)$ which is continuous on K and analytic in the interior of K may be approximated uniformly on K by functions analytic on a prescribed open set G containing K in its interior.

From the point of view of differential equations, the class of analytic functions is merely the class of solutions of the homogeneous first-order linear elliptic differential equation with constant complex coefficients:

$$\frac{\partial u}{\partial \bar{z}} = 0,$$

where $\partial/\partial\bar{z}$ is the classical Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$$

in the plane. The existence of theorems of the Walsh-Lavrentiev-Mergelyan type for the Cauchy-Riemann operator raises the question of possible generalizations of such results for solutions of general

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