ON THE CENTRAL LIMIT THEOREM IN R_k

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Let $\xi_1, \xi_2, \cdots, \xi_n, \cdots$, be a sequence of independent and identically distributed random vectors in R_k with finite second order moments. Let $\eta_n = (\xi_1 + \cdots + \xi_n)n^{-1/2}$ and let $P_n(A) = P[\eta_n \in A]$. Let η denote a random vector in R_k which is normally distributed and whose moments of the first two orders are identical with those of ξ_1 and let $P(A) = P[\eta \in A]$. Then, by the central limit theorem in R_k, P_n weakly converges to P. A question that arises naturally here is an investigation of the error of approximation $P_n - P$. This problem has been thoroughly investigated in the case $k=1$ (cf. [**3**; **4**; **5**] and also the survey [**6**] where a complete set of references is given). For $k>1$, Bergström [**1**; **2**] obtained bounds on the error

$$\sup_{x \in R_k} | F_n(x) - \Phi(x) |$$

where F_n, Φ are the distribution functions of η_n and η respectively. Esseen [**5**] gave similar bounds for the error $|P_n(A) - P(A)|$, when A is a sphere with centre at the origin. The object of this study is to investigate the error $\Delta_n(A) = P_n(A) - P(A)$ for a very wide class of sets—namely the class of all convex subsets of R_k.

2. **Notation and preliminaries.** Let $\xi_1 = (\xi_1^{(1)}, \cdots, \xi_1^{(k)})$. We suppose that $E\xi_1^{(j)} = 0$ for $j = 1, 2, \cdots, k$, and that the variance covariance matrix of ξ_1, to be denoted by V, is nonsingular. We use the following notation for denoting the moments and cumulants of ξ_1:

$$\beta_s = \sum_{j=1}^{k} E | \xi_1^{(j)} |^s;$$

the cumulant of order $(s_1, s_2, \cdots, s_k)$ will be denoted by $\lambda_1^{s_1} \cdot \lambda_2^{s_2} \cdot \cdots \cdot \lambda_k^{s_k}$. Let $f(t)$ denote the characteristic function of ξ_1. Then the characteristic function of η_n is $[f(tn^{-1/2})]^n$. Let the polynomials $\tilde{P}_j(w)$ in the vector $w = (w_1, \cdots, w_k)$ be defined by the formal identity:

$$(1) \quad \exp\left\{ \sum_{j=3}^{\infty} \frac{1}{j!} (\lambda_1 w_1 + \cdots + \lambda_k w_k)^j n^{-(j-2)/2} \right\} = \sum_{j=0}^{\infty} n^{-j/2} \tilde{P}_j(w).$$

(Here the λ's represent the cumulants of ξ_1.) Let the functions $\tilde{P}_j(-\phi)$, $\tilde{P}_j(-\Phi)$ for $j = 0, 1, \cdots$, be defined as follows: