TOPOLOGICAL EQUIVALENCE OF A BANACH SPACE WITH ITS UNIT CELL

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Several years ago [8] we proved that Hilbert space is homeomorphic with both its unit sphere $\{x: \|x\| = 1\}$ and its unit cell $\{x: \|x\| \leqq 1\}$. Later [9] we showed that in every infinite-dimensional normed linear space, the unit sphere is homeomorphic with a (closed) hyperplane and the unit cell with a closed halfspace. It seems probable that every infinite-dimensional normed linear space is homeomorphic with both its unit sphere and its unit cell, but the question is unsettled even for Banach spaces. Corson [4] has recently proved that every $\aleph_0$-dimensional normed linear space is homeomorphic with its unit cell. In the present note, we establish the same result for a class of infinite-dimensional Banach spaces which is *believed* to include *all* such spaces. It is *proved* to include every infinite-dimensional Banach space which is reflexive, or admits an unconditional basis, or is a separable conjugate space, or is a space CM of all bounded continuous real-valued functions on a metric space M.

We employ the following tools:

(1) If E and F are Banach spaces and u is a continuous linear transformation of E onto F, then there exist a constant $m \in]0, \infty[$ and continuous mapping v of F into E such that $uvx = x$, $vrx = rvx$, and $\|vx\| \leqq m\|x\|$ for all $x \in F$ and $r \in R$ (the real number space). If G is the kernel of u and $hy = (uy, vuy - y) \in F \times G$ for each $y \in E$, then h is a homeomorphism of E onto $F \times G$. Let $\|(p, q)\| = \max(\|p\|, \|q\|)$ for all $(p, q) \in F \times G$, and let $\xi y = (\|y\|/\|hy\|)hy$ for all $y \in E$. Then ξ is a homeomorphism of E onto $F \times G$ which carries the unit cell of E onto that of $F \times G$.

(2) If S is a closed linear subspace of a Banach space E, then E is homeomorphic with the product space $(E/S) \times S$ and the unit cell of E is homeomorphic with the unit cell of this product space (with respect to any norm compatible with the product topology).

(3) In each infinite-dimensional normed linear space, the unit cell is homeomorphic with a closed halfspace.

(4) If Q is an open halfspace in an infinite-dimensional normed linear space and p is a point in the boundary of Q, then $Q \cup \{p\}$ is homeomorphic with Q.

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