DERIVATIONS OF COMMUTATIVE BANACH ALGEBRAS

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In [2] Singer and Wermer showed that a bounded derivation in a commutative Banach algebra $\mathfrak{A}$ necessarily maps $\mathfrak{A}$ into the radical $\mathfrak{R}$. They conjectured at this time that the assumption of boundedness could be dropped. It is a corollary of results proved below that if $\mathfrak{A}$ is in addition regular and semi-simple, this is indeed the case.

What is actually proved here is that under the above hypotheses, if D is a derivation of $\mathfrak{A}$ into $C(\Phi_{\mathfrak{A}})$,[2] $\Phi_{\mathfrak{A}}$ the structure space of $\mathfrak{A}$, then D is a bounded operator from $\mathfrak{A}$ to $C(\Phi_{\mathfrak{A}})$. The topologies are the norm topology in $\mathfrak{A}$ and the sup norm topology in $C(\Phi_{\mathfrak{A}})$. An application of the closed graph theorem shows that if D maps $\mathfrak{A}$ into itself, D must be a bounded operator in $\mathfrak{A}$, hence by the Singer, Wermer theorem, $D=0$.

If $\mathfrak{A}$ is regular but not semi-simple, then it follows from the above that D will map $\mathfrak{A}$ into $\mathfrak{R}$ provided that D maps $\mathfrak{R}$ into $\mathfrak{R}$. This the author can verify only if $\mathfrak{R}$ is nilpotent.

In what follows $\mathfrak{A}$ will always denote a regular, commutative, semi-simple Banach algebra with norm $\|\cdot\|$. Applying the Gelfand isomorphism we will identify $\mathfrak{A}$ and the corresponding subalgebra of $C(\Phi_{\mathfrak{A}})$. For convenience we also will assume $\mathfrak{A}$ possesses an identity. It is easily seen that this doesn't affect the generality of the results.

Let $\mathfrak{M}_\phi$ be a maximal ideal of $\mathfrak{A}$, and ϕ the corresponding point in $\Phi_{\mathfrak{A}}$. It is noted in [2] that there exists a derivation D of $\mathfrak{A}$ into some semi-simple extension $\mathfrak{B}$ of $\mathfrak{A}$ iff $\mathfrak{M}_\phi^2 \neq \mathfrak{M}_\phi$ for some maximal ideal $\mathfrak{M}_\phi$. In fact $\mathfrak{B}$ may be taken to be $B(\Phi_{\mathfrak{A}})$, the ring of bounded complex functions on $\Phi_{\mathfrak{A}}$. For if this condition is satisfied, following Singer and Wermer, we define by Zorn's Lemma a nontrivial linear functional f_ϕ on $\mathfrak{A}$ which annihilates $\mathfrak{M}_\phi^2$ and the identity. If we define D by

$$\begin{aligned} Dx(\phi') &= 0, \qquad \phi' \in \Phi_{\mathfrak{A}}, \qquad \phi' \neq \phi, \qquad x \in \mathfrak{A}, \\ Dx(\phi) &= f_\phi(x), \end{aligned}$$

it is easily seen that D is a derivation of $\mathfrak{A}$ into $B(\Phi_{\mathfrak{A}})$. D is in general unbounded, but if $\overline{\mathfrak{M}_\phi^2} \neq \mathfrak{M}_\phi$, f_ϕ, and consequently D, may be chosen (via the Hahn-Banach Theorem) to be bounded. Modifying the

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[2] $C(\Phi_{\mathfrak{A}})$ denotes the algebra of continuous complex functions on the space $\Phi_{\mathfrak{A}}$.