THE MANIFOLD SMOOTHING PROBLEM[1]

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The Schoenflies Theorem in n dimensions has been proved by both Marston Morse [4] and Morton Brown [1] subject to the shell hypothesis [4]. Morse's proof leads to C^m-diffeomorphisms. We now prove the following Schoenflies Theorem for polyhedra without the shell hypothesis.

THEOREM 1.[2] *Let P^{n-1} be a combinatorial $(n-1)$-sphere in a euclidean n-space E^n, and let N be an arbitrary neighborhood of P^{n-1}. Then E^n can be mapped onto itself by a homeomorphism h which is a C^∞-diffeomorphism on E^n-N and which maps P^{n-1} onto a euclidean $(n-1)$-sphere S^{n-1}.*

The proof commences with a modification of a procedure due to H. Noguchi [5] yielding an ϵ-isotopy of E^n carrying P^{n-1}, on D^n, into a polyhedron Q^{n-1}, admitting a transverse vector field. A neighborhood of Q^{n-1} is fibred by C^∞—$(n-1)$-spheres, which permits a completion of the proof with the aid of Morse's methods [4]. His exceptional interior point can be relegated to N. The proof is inductive, requiring a partial assumption of Theorem 1 in the next lower dimension.

COROLLARY. *Given a $\delta>0$, E^n admits a δ-isotopy h_t $(0\leqq t\leqq 1)$ such that* (1) *h_t is the identity on the unbounded component of E^n-N,* (2) *$h_t(P^{n-1})\subset D^n$ $(t>0)$ and* (3) *$h_t(P^{n-1})$ is a C^∞-$(n-1)$-sphere $(t>\delta)$.*

We will call a combinatorial n-manifold *smoothable* or *nonsmoothable* according as it is or is not compatible with a differentiable structure. The known nonsmoothable manifolds include a K^8 due to Milnor [3] and a K^{10} due to Kervaire [2]. The latter is *strongly nonsmoothable*, in the sense that the topological manifold it covers, $M^{10}=|K^{10}|$, can not carry a differentiable structure, either compatible or incompatible with K^{10}.

A piecewise differentiable imbedding of a K^m in a differentiable n-manifold M^n means a homeomorphism $h: K^m\rightarrow M^n$, where h is differentiable of maximal rank on each closed simplex of K^m.

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[2] A sharpening of this theorem appears in Proc. Nat. Acad. Sci. U.S.A. vol. 47, (1961) pp. 328–330.