INTEGRATION WITH RESPECT TO OPERATOR-VALUED FUNCTIONS

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1. Introduction. Let J be a compact subinterval of the real line. N. Wiener [7] has introduced the Banach algebra $W_p(J)$ of all complex-valued functions f such that $V_p(f) \neq \infty$, where

$$V_p(f) = \sup\left(\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|^p \right)^{1/p},$$

the supremum being taken over all finite partitions of J (see §7). We shall construct a family of continuous homomorphisms of the Banach algebra $W_p(J)$; this connects with the theory of multipliers of Fourier series (see §4). Our basic problem is to integrate (in the uniform operator-topology) with respect to functions that are not of bounded variation.

Given a fixed measure space $(a, \mathfrak{A}, \mu)$, let $\mathfrak{E}_r$ denote the Banach algebra of all continuous endomorphisms of $L_r(a, \mathfrak{A}, \mu)$; the relation $1 < r < \infty$ is implied throughout. Let E_r be a function on J which assumes its values in $\mathfrak{E}_r$, and let f belong to the class $D(J)$ of all simply-discontinuous,[2] complex-valued functions. The following expression

$$(1) \qquad (\mathfrak{E}_r) \int f(\lambda) \cdot dE_r(\lambda)$$

will denote what T. H. Hildebrandt [1, p. 273] calls the "*modified Stieltjes integral*"; it is the limit of a certain net of Stieltjes sums (this net is directed as in the Pollard-Moore integral [1, p. 269]). The word "limit" here implies convergence in the norm-topology of $\mathfrak{E}_r$. It is not hard to show that the integral (1) converges when E_r is of bounded variation[3]; this situation is most familiar in the case $r = 2$, when E_r is a resolution of the identity in the Hilbert space $L_2(a, \mathfrak{A}, \mu)$. Henceforth, we will allow the possibility that E_r not be of bounded variation (this possibility becomes a fact in Theorem D below).

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[2] That is, having on J at most discontinuities of the first kind.

[3] In the sense of Hille-Phillips [2, p. 59]. Bounded variation is a less restrictive condition than the bounded semi-variation hypothesis required in certain integration theories (e.g., Bartle's article in the Studia Math. vol. 15 (1956) pp. 337–352).