ON THE BRAID GROUPS OF E^2 AND S^2

BY EDWARD FADELL[1] AND JAMES VAN BUSKIRK

Communicated by Deane Montgomery, December 11, 1960

1. **Introduction.** The purpose of this paper is to outline a new proof that the geometric braid group on the plane E^2 is isomorphic to the algebraic braid group on E^2 and also to compute the geometric braid group on the sphere S^2. The definition of geometric braid group (see §3) employed is a recent one due to R. H. Fox [3]. It turns out that the geometric braid group on S^2 is just the classical braid group on E^2 with a single additional relation obtained from the simple geometric fact that a simple closed curve on S^2 bounds *two* discs. In [2], it is shown that an associated class of fiber spaces arising naturally from the situation gives information about certain homotopy groups which in turn gives information about the geometric braid groups of certain manifolds. This point of view is pursued in this paper and the central geometric tool here is the fact that the second homotopy group of certain configuration spaces is trivial.

2. **Configuration spaces.** Let M denote a manifold and $x_1, \cdots, x_m$ a fixed set of m mutually distinct points. Set $F_{m,n}(M)$ = the space of n-tuples $(p_1, \cdots, p_n)$ such that $p_i \in M-(x_1 \cup \cdots \cup x_m)$ and $p_i \neq p_j$ if $i \neq j$. Consider $\pi: F_{m,n} \rightarrow F_{m,n-r}$, $n > r$, $m \geqq 0$, given by $\pi(p_1, \cdots, p_n) = (p_{r+1}, \cdots, p_n)$.

THEOREM [2]. *$\pi: F_{m,n} \rightarrow F_{m,n-r}$ is a locally trivial fiber space with fiber $F_{m+n-r,r}$.*

THEOREM [2]. *If $M = E^2$ (Euclidean 2-space), then $\pi_i(F_{0,n}) = 0$ for $i \geqq 2$, $n \geqq 1$.*

THEOREM. *If $M = S^{k-1}$ ($(k-1)$-sphere), $k \geqq 3$, then the fiber space $\pi: F_{0,3} \rightarrow S^{k-1}$ is fiber homotopy equivalent to the bundle $V_{k,2} \rightarrow S^{k-1}$, where $V_{k,2}$ is the Stiefel manifold of orthogonal 2-frames in k-space.*

COROLLARY. *If $M = S^2$, $\pi_1(F_{0,3})$ is cyclic of order 2 and $\pi_2(F_{0,n}) = 0$ for $n \geqq 3$.*

3. **The geometric braid groups.** Consider $F_{0,n} = F_{0,n}(M)$, where M is a manifold. Then the full symmetric group Σ^n acts freely on $F_{0,n}$ by permuting coördinates. Let $B_{0,n} = F_{0,n}/\Sigma^n$ and $p: F_{0,n} \rightarrow B_{0,n}$ be the associated covering space with fiber Σ^n.

[1] This research was supported in part by the National Science Foundation under Grant NSF G-11297.