

## RESEARCH PROBLEMS

1. Richard Bellman: *Generalized exponentials and Baker-Hausdorff-Campbell series.*

Consider the equation

$$(1) \quad u = 1 + T(xu),$$

where  $T$  is a linear transformation and  $x$  and  $u$  are elements of a suitable space. As pointed out by Baxter, if  $T$  enjoys the following generalized "integration by parts" formula

$$(2) \quad T(xT(y)) + T(yT(x)) = T(x)T(y)$$

then  $u = e^{T(x)}$  is a solution of (1), and the solution under appropriate assumptions.

Generally, let us write  $u = E(x)$  to denote the solution of (1), a *generalized exponential*, and introduce a *generalized bracket symbol*

$$(3) \quad [x, y; T] = T(xT(y)) + T(yT(x)) - T(x)T(y).$$

It is easy to verify that this enjoys the Friedrichs-Magnus property

$$(4) \quad [x + x', y + y'; T] = [x, y; T] + [x', y'; T].$$

One would suspect in view of the foregoing remarks that this new bracket symbol plays the same role in the study of the function  $E(x)$  that the classical commutator,  $[A, B] = AB - BA$ , plays in the study of the matrix exponential.

We would thus expect formulas of the type

$$(5) \quad E(x + y) = E(x)E(y)E([x, y; T]) \cdots,$$

where the further terms contain iterations of the bracket operation, and equivalently, a generalized Baker-Hausdorff-Campbell formula

$$(6) \quad E(x)E(y) = E(x + y + [x, y; T]/2 + \cdots).$$

Finally, there should be an associated generalized Lie algebra.

Do formulas of the above type exist, and how does one obtain them?

2. Robert J. Aumann: *Extending an order.*

Let  $Z^n$  denote the set of lattice points in euclidean  $n$ -space,  $I^n$  the set of points in  $Z^n$  all of whose coordinates are 0 or 1 (i.e., the vertices of the unit cube). Let  $>$  be a total order on  $I^n$  which is "consistent" in the sense that for  $x, y, z, x+z, y+z \in I^n$ ,  $x > y$  implies  $x+z > y+z$  (ordinary vector addition is meant). Is it always possible to extend  $>$  to a "consistent" order on  $Z^n$ ?