FIXED POINTS OF MULTIPLE-VALUED TRANSFORMATIONS

F. B. FULLER

A *multiple-valued transformation* T from a space X to a space Y is a function assigning to each point x of X a nonempty closed subset $T(x)$ of Y. The *graph* of T comprises those points (x, y) in the topological product XY for which y belongs to $T(x)$. All spaces to be considered shall be compact metric and all transformations T shall be upper semi-continuous, meaning that their graphs are closed, hence compact, subsets of XY.

When the domain X and the range Y of T coincide, a *fixed point* of T is defined to be a point x which belongs to its image set $T(x)$. The fixed points correspond to those points in the product XX which belong to the intersection of the graph of T with the diagonal of XX. In the special case that X is an orientable n-manifold the diagonal carries an n-cycle D. If, in addition, each neighborhood of the graph contains a representative cycle of an n-cycle homology class Γ whose intersection number with D is not zero, then the diagonal must meet the graph of T, so that under the assumptions made at least one fixed point must exist. To each n-cycle class Γ having representative cycles in each neighborhood of the graph corresponds an endomorphism $T_{*\Gamma}$ of the homology group $H(X)$ of X, determined as follows: starting with any p-cycle γ, form in the product XX the upright cylinder $\gamma \times X$, intersect the cylinder with Γ and project the intersection laterally into X to obtain finally $T_*(\gamma)$. If the homology of X uses a field as coefficient group, then the endomorphisms $T_{*\Gamma}$, constitute a vector space $*(T)$. For a single-valued transformation τ, the space $*(\tau)$ comprises scalar multiples of the conventional endomorphism τ_* induced by τ. The definition of $*(T)$, here described for manifolds only, has been extended to an arbitrary A.N.R., using singular homology, by Lefschetz [8] and to an arbitrary compact metric space, using Čech homology, by O'Neill [11].

The Lefschetz fixed point theorem, extended to multiple-valued transformations, assumes the following form: Let X be an A.N.R. and let T be an upper semi-continuous multiple-valued transformation of X into itself. Then either T has a fixed point or else the equa-

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