ON A PROPERTY OF FUNCTIONS WITH NON-NEGATIVE DERIVATIVES AT THE ORIGIN AND ITS APPLICATION[1]

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Communicated by J. L. Doob, October 7, 1960

Functions which have non-negative derivatives at the origin play an important role in the theory of probability. The moment generating functions of symmetric distributions are the most common examples of such functions. In the present note we first give a theorem on the properties of such functions. Finally we consider an interesting application of this theorem in deducing an analytic decomposition of the Poisson distribution. For this purpose we require the following definition:

DEFINITION. Let $g(x)$ be a real-valued function of the real variable x which has finite derivatives of all orders

$$g^{(k)}(0) = \left[\frac{d^k}{dx^k} g(x)\right]_{x=0}, \qquad k = 1, 2, 3, \cdots,$$

at the point $x=0$. We then define formally the function

$$G(z) = g(0) + \sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} z^k \qquad (z \text{ complex})$$

as a function of the complex variable z ($z=x+iy$, x and y being both real) and call $G(z)$ the adjoint function of $g(x)$. In general, the function $G(z)$ may not exist, except at the point $z=0$ and we note that $G(0)=g(0)$. The adjoint function $G(z)$ is said to be regular in the circle $|z|<R$ $(R>0)$ about the point $z=0$ if and only if the power series $g(0)+\sum_{k=1}^{\infty}(g^{(k)}(0)/k!)z^k$ converges in the same circle $|z|<R$. We can now formulate the following theorem:

THEOREM 1. *Let* $\{g_n(x)\}$, $n=1, 2, \cdots$, *be a countable sequence of real valued functions of the real variable* x *such that* $g_n(x)\geqq 1$ *for* $0<x<\delta$ $(\delta>0)$ *and further each* $g_n(x)$ *has non-negative derivatives of all orders at the point* $x=0$. *Let* $\{\alpha_n\}$ *be a sequence of positive numbers such that* $0<\alpha_0\leqq\alpha_n\leqq\alpha_0'$, $n=1, 2, \cdots$. *Let* $g(x)$ *be a function of the real variable* x *having finite derivatives of all orders at the point* $x=0$ *such that the corresponding adjoint function* $G(z)$ *is regular in a circle* $|z|<R$ $(R>0)$ *about the point* $z=0$ *and has no zeros inside this circle. Suppose that the*

[1] This work was supported by the National Science Foundation through grant NSF-G-9968.