A REMARK ON PICARD'S THEOREM

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1. In a recent manuscript K. Matsumoto has proved that there exists no general Picard theorem for functions meromorphic outside a set of logarithmic capacity zero. More precisely: given a closed set E of logarithmic capacity zero, then there exists another closed set F of capacity zero and a function $f(z)$, meromorphic in the complement of F and with essential singularities at all points of F, such that $f(z)$ omits the set E. This result indicates that there is no Picard theorem at all for any perfect set. We shall here show that there still is an essential problem left, by giving an example of a set F for which a strong Picard's theorem holds.[1]

THEOREM. *There exists a linear closed set of positive capacity such that every function $f(z)$, meromorphic outside F and omitting 4 values is rational.*

2. We construct F as a Cantor set on $(0, 1)$ where the successive ratios ξ_ν decrease and satisfy the condition

$$\lim_{\nu=\infty} \frac{\log \xi_\nu}{\log \nu} = -\infty. \tag{2.1}$$

Since F is of positive capacity if and only if

$$\sum_1^\infty \frac{\log \xi_\nu^{-1}}{2^\nu} < \infty,$$

ξ_ν can be chosen so that cap $(F) > 0$. Let $f(z)$ be meromorphic outside F and assume that $f(z) \neq a_1, a_2, a_3, a_4$. Define $f_\nu = (1 - fa_\nu)(f - a_\nu)^{-1}$.

For the functions f_ν, the following lemmas hold. M denotes constants only depending on a_ν.

LEMMA 1. *If f_ν is holomorphic in $\rho \leqq |z-a| \leqq 2\rho$ and $|f_\nu(z_0)| \leqq M$, $|z_0 - a| = 3\rho/2$, then $|f_\nu(z)| \leqq M$ for all z, $9\rho/8 \leqq |z-a| \leqq 15\rho/8$.*

PROOF. This is Schottky's theorem.

LEMMA 2. *If $f_\nu(z)$ is holomorphic in $\rho \leqq |z-a| \leqq K\rho$, $K \geqq 2$, and $|f_\nu(z)|$ is $< M$, then the circle $|z-a| = K^{1/2}\rho$ is mapped on a set of diameter $< MK^{-1/2}$.*

[1] This type of strong Picard theorem was used by O. Lehto, *A generalization of Picard's theorem*, Ark. Mat. vol. 3 (1958) p. 495.