POTENTIALS FOR DENUMERABLE MARKOV CHAINS

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Probabilistic generalizations of classical potential theory have been worked out by J. L. Doob [1] and G. A. Hunt [2]. Specialized to discrete processes, this provides a potential theory for transient Markov chains, analogous to the theory of the Newtonian potential. We provide a unified potential theory for *all* denumerable Markov chains; as applied to recurrent chains, the theory generalizes classical results on logarithmic potentials. We denote by P the transition matrix of a Markov chain with states the integers.

DEFINITION. A function of states f is called a *right potential charge* if it is integrable with respect to a given non-negative superregular measure α ($\alpha P \leqq \alpha$), and if the limit $g = \lim_n (I+P+ \cdots +P^n)f$ exists. Then g is called a *right potential* with charge f. If μ is a finite measure and $\nu = \lim_n \mu(I+P+ \cdots +P^n)$ then ν is called a *left potential* with *charge* μ.

The mapping $f \to \mu$, with $\mu_i = f_i \alpha_i$ is an isomorphism preserving all the interesting properties. E.g., f is a right charge for P if and only if μ is a left charge for the so-called reverse chain. Since the reverse chains include all Markov chains, we automatically obtain for each theorem about functions a dual theorem about measures.

In the transient case potentials can be represented by means of a positive potential operator $G = I+P+P^2+ \cdots$, as $g = Gf$ or $\nu = \mu G$.

In the recurrent case we show that $\alpha f = 0$ is a necessary condition, or dually, μ must have total measure 0.

In this case we have dual positive operators

$$G_{ij} = \lim_n \, [N_{ii}^{(n)} \alpha_j / \alpha_i - N_{ij}^{(n)}] \quad \text{and} \quad C_{ij} = \lim_n \, [N_{jj}^{(n)} - N_{ij}^{(n)}],$$

where $N_{ij}^{(n)}$ is the mean number of times that the process is in j in the first n steps, starting at i. It is shown that one exists if and only if the other does, and either is equivalent to the existence of the limiting probabilities ${}^0\lambda_j = \lim_n \sum_k P_{ik}^n \, {}^0h_{kj}$ for a fixed state 0. Here ${}^0h_{kj}$ is the probability, starting at k, that j is reached before 0. If any one of these conditions is fulfilled, we say that the chain is *normal.*

All ergodic (positive recurrent) chains are normal.

We know of no chain that fails to be normal, but we cannot prove that all null chains are normal. We have verified that important