TOTAL POSITIVITY, ABSORPTION PROBABILITIES AND APPLICATIONS

BY SAMUEL KARLIN

Communicated by I. J. Schoenberg, August 24, 1960

A function $K(x, y)$ of two real variables ranging over linearly ordered one-dimensional sets X and Y respectively is said to be *totally positive of order* $r(TP_r)$ if for all $1 \leqq m \leqq r$, $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$ $(x_i \in X;\ y_j \in Y)$, we have the inequalities

$$(1)\quad K\begin{pmatrix} x_1, x_2, \cdots, x_m \\ y_1, y_2, \cdots, y_m \end{pmatrix} = \begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) & \cdots & K(x_1, y_m) \\ K(x_2, y_1) & K(x_2, y_2) & \cdots & K(x_2, y_m) \\ \vdots & \vdots & & \vdots \\ K(x_m, y_1) & K(x_m, y_2) & \cdots & K(x_m, y_m) \end{vmatrix} \geqq 0.$$

Typically, X is an interval of the real line or a countable set of discrete values on the real line; similar for Y.

A related concept is that of sign regularity. A function $K(x, y)$ *is sign regular of order* r, if for every $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$ $(x_i \in X;\ y_j \in Y)$ and m, $1 \leqq m \leqq r$,

$$(2)\qquad \operatorname{sign} K\begin{pmatrix} x_1, x_2, \cdots, x_m \\ y_1, y_2, \cdots, y_m \end{pmatrix} = \epsilon_m$$

depends on m alone.

An important specialization occurs if a TP_r function may be written as a function $K(x-y)$ of the difference of x and y where x and y traverse the real line (or the set of integers); $K(u)$ is then said to be a *Pólya frequency density* of order $r(PF_r)$. In this case we assume $K(u)$ is integrable with respect to Lebesgue measure.

The theory of totally positive functions has been extensively applied in several domains of mathematics, statistics and mechanics. The notion of total positivity is a common thread running through many disciplines. They derive interest in their intrinsic relevance to the theory of stochastic processes of diffusion type [7]; in their natural occurrence in applications to statistics [5; 6] and mechanics [1]; and in their elegant structural properties [8].

Examples of totally positive functions appear abundantly and naturally in different contexts. Most of the standard statistical distributions depending on a parameter such as the exponential family,