THE CHARACTERIZATION OF FUNCTIONS ARISING AS POTENTIALS

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1. **Introduction.** The purpose of this note is to announce results giving the characterization of classes of functions arising by fractional integration. We consider two fractional integral operators I_α and J_α, defined for a suitable class of functions on E_n as follows:

$$I_\alpha(f)^\frown(x) = |x|^{-\alpha} f^\frown(x), \qquad 0 < \alpha < n,$$
$$J_\alpha(f)^\frown(x) = (1 + |x|^2)^{-\alpha/2} f^\frown(x), \qquad 0 < \alpha.$$

The symbol $^\frown$ denotes the Fourier transform.

The integral I_α is a well-known Riesz potential, while the integral J_α is a modification of it, the so-called "Bessel potential." The local behavior of I_α and J_α are equivalent, but the global behavior of J_α is more tractable since $J_\alpha f = K_\alpha^* f$, where $K_\alpha \geqq 0$, and $K_\alpha \in L^1(E_n)$.

We denote by L^p_α the class of functions f of the form $f = J_\alpha(\phi) = K^*_\alpha \phi$, where $\phi \in L^p(E_n)$. We shall always make the restriction $1 < p < \infty$. The classes L^p_α and the operators J_α have been studied by several authors; see [1] for the L_2 theory, and [2] for the L^p theory. We seek to characterize the functions $f \in L^p_\alpha$ in terms of their "smoothness," i.e., in terms of the smallness of $f(x+y) - f(x)$.

We recall first a useful fact. If $f \in L^p_\alpha$, $\alpha \geqq 1$, then $f \in L^p_{\alpha-1}$ and $\partial f/\partial x_k \in L^p_{\alpha-1}$, $k = 1, \cdots, n$, and conversely. Thus in many cases it is sufficient to restrict our attention to $0 < \alpha < 1$.

Our characterization will be in terms of the "functional" $\mathfrak{D}_\alpha$

$$\mathfrak{D}_\alpha(f)(x) = \left(\int_{E_n} \frac{|f(x-y) - f(x)|^2}{|y|^{n+2\alpha}}\, dy\right)^{1/2}, \qquad 0 < \alpha < 1, \tag{1}$$

and its variants.

2. **Main results.**

THEOREM 1. *Let* $0 < \alpha < 1$, $2n/(n+2\alpha) < p < \infty$. *Then* $f \in L^p_\alpha$ (*i.e.*, $f = J_\alpha(\phi)$, $\phi \in L^p$) *if and only if* (a) $f \in L^p$, *and* (b) $\mathfrak{D}_\alpha(f) \in L^p$. *Also*

$$B_{\alpha,p}\|\phi\|_p \leqq \|\mathfrak{D}_\alpha(f)\|_p + \|f\|_p \leqq A_{\alpha,p}\|\phi\|_p.$$

REMARKS. (i) The restriction $2n/(n+2\alpha) < p$ is essentially necessary. If $p < 2n/(n+2\alpha)$, there exists $\phi \in L^p$, so that $f = J_\alpha(\phi)$ is not locally in L^2 and so that $\mathfrak{D}_\alpha(f)(x) = \infty$, all x.