# A GENERALIZATION OF $H$-SPACES[1]

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**1. Introduction.** An $H$-space is a topological space $S$ with a continuous multiplication $f: S\times S\to S$, $f(x, y)=x\cdot y$, *having a two-sided unit* $e$: thus $e\cdot x=x$, $x\cdot e=x$ for all $x$ in $S$. We shall consider spaces $S$ with a more general type of product: namely, instead of assuming a two-sided unit $e$, we assume only

(i) $e\cdot x=x$ for all $x$.

(ii) There is a continuous map $\sigma: S\to S$ such that $x\cdot\sigma(x)=x$ for all $x$. Thus if $\sigma(x)=e$ for all $x$, we have an $H$-space.

A general class of such spaces $S$ is constructed as follows: let $G$ be a topological group, $\sigma$ a continuous endomorphism, $K$ a closed subgroup of $G$ *contained in* (not necessarily equal to) the fixed point set of $\sigma$; let $S=G/K$, the space of left cosets, and define a product in $S: f(g_1K, g_2K)=g_1\sigma(g_1^{-1})g_2K$. Another way of looking at this product is the following: since $G$ acts on the left on $G/K$, any continuous map $q: G/K$ into $G$, defines a product on $G/K$ by $f(g_1K, g_2K)=q(g_1K)g_2K$. In the above situation we have taken the map $q(gK)=g\sigma(g^{-1})$. The product then satisfies (i) and (ii) above, with $\sigma(gK)=\sigma(g)K$. Note that if $\sigma$ maps all of $G$ onto the identity element, then $S=G$ and the product is just the product in $G$. We also remark that if $q$ is any *cross-section* of $G/K$ into $G$ (i.e., $\pi q=$identity map of $G/K$ where $\pi: G\to G/K$, $\pi(g)=gK$) and $q(eK)=e$, the identity element of $G$, then the multiplication $g_1K\cdot g_2K=q(g_1K)g_2K$ makes $G/K$ an $H$-space. Such a $q$ is obtained, for instance, if $\sigma^2=\sigma$, $K=\sigma(G)$, and $q=g\sigma(g^{-1})$. We shall be more interested, however, in the case $\sigma^2=I$, the identity map: if, further, $K$ contains the identity component of the fixed point set of $\sigma$, then $S=G/K$ is called a symmetric space. The cohomology algebra, with *real* coefficients, of symmetric spaces of compact Lie groups $G$, is completely known (see [1; 2]); however, with coefficients a field of characteristic $p>0$ less is known and our results when specialized to this case, seem to be new. On taking $G=\mathrm{SO}(n+1)$ the rotation group, $K=\mathrm{SO}(n)$, $G/K=S^n$ and $n$ *odd*, the product in the sphere $S^n$ is essentially the same[2] as one defined by Hopf (in a purely geometric way) in his paper [3] which introduced the subject of $H$-spaces.

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[2] Actually Hopf's product, as is easy to see, is $(g_1K, g_2K)\to g_1\sigma(g_1^{-1})\sigma(g_2)K$, but study of this latter product is equivalent to study of the former.