ON A PROBLEM OF KLEENE'S

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THEOREM I.[1] *The class of functions of hyperdegree strictly less than* $\mathbf{0}'$ *provides a basis for the predicate* $(E\alpha)(x)\overline{T}_1^1(\bar{\alpha}(x), a, a)$, *and hence for all predicates which belong to* Σ_1^1.

This theorem settles a problem left open by Kleene in [**4**]. To prove it we observe that Theorem XXVI of [**3**] relativises uniformly to an arbitrary function α (see Theorem XXVII of [**3**]). Thus there is a recursive $K(u, v)$ such that:

(i) $(\alpha)(E\beta)(x)K(\bar{\alpha}(x), \bar{\beta}(x))$;

(ii) $(\alpha)(\beta)_{\beta\in HA(\alpha)}(\bar{x})K(\bar{\alpha}(x), \bar{\beta}(x))$,

where $HA(\alpha)$ denotes the class of functions hyperarithmetic in α.

Suppose a satisfies the predicate $(E\alpha)(x)\overline{T}_1^1(\bar{\alpha}(x), a, a)$; then, by (i), there exist functions α, β such that

(A) $$(x)\overline{T}_1^1(\bar{\alpha}(x), a, a) \quad \& \quad (x)K(\bar{\alpha}(x), \bar{\beta}(x)).$$

And we can construct such functions recursively in O (cf. 5.5 (5) of [**5**]). But if $O\in HA(\alpha)$ then also $\beta\in HA(\alpha)$, which would contradict (ii). Hence there is an α of hyperdegree strictly less than $\mathbf{0}'$ such that $(x)\overline{T}_1^1(\bar{\alpha}(x), a, a)$; and this proves the theorem.[2]

By an obvious elaboration of the above argument we can construct, recursively in O, an infinite sequence of non-hyperarithmetic functions α_i such that $\boldsymbol{\alpha}_1 \not< \boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_2 \not< \boldsymbol{\alpha}_0 \cup \boldsymbol{\alpha}_1, \cdots$ (where bold face type denotes a hyperdegree). Thus we can prove

COROLLARY 1. *There are infinitely many distinct hyperdegrees lying between* $\mathbf{0}$ *and* $\mathbf{0}'$.

COROLLARY 2.[3] *If a* Π_1^1 *set of axioms for second-order arithmetic has an* ω*-model, then it has an* ω*-model whose functions are all of hyperdegree strictly less than* $\mathbf{0}'$.

[1] For notations used see [**2**; **3**; **6**]; in particular we use boldface type for hyperdegrees. $\mathbf{0}'$ is the hyperdegree of O.

[2] G. Kreisel points out that a similar construction may be used to prove a result of J. R. Shoenfield's (*Degrees of models*, Amer. Math. Soc. Notices vol. 6 (1959) p. 530): the functions whose degree is strictly less than the degree $\mathbf{0}'$ provide a basis for Σ_1 predicates in which the existential function quanfiier is bounded by a given recursive function. I am also indebted to Kreisel for suggesting Corollary 2 below.

[3] By a "Π_1^1 set of axioms" we mean a set of formulae whose Gödel numbers form a Π_1^1 set. An "ω-model" is a model which is standard with respect to the natural numbers.