COMBINATORIAL TOPOLOGY OF AN ANALYTIC FUNCTION ON THE BOUNDARY OF A DISK

BY CHARLES J. TITUS

Communicated by Edwin Moise, August 12, 1960

Preliminaries. A complex valued function $\zeta(t)$ defined on an oriented circle S of circumference c, t the usual distance parameter, $0 \leqq t < c$, is a regular representation if it possesses a continuous non-vanishing derivative $\zeta'(t)$. An image point ζ_0 is a *simple crossing point* if there exist exactly two distinct numbers t_0' and t_0'' such that $\zeta(t_0') = \zeta(t_0'') = \zeta_0$ and if the tangents $\zeta'(t_0')$ and $\zeta'(t_0'')$ are linearly independent. A regular representation is *normal* (Whitney) if it has a finite number of simple crossing points and has for every other image point ζ but one preimage point t. A pair of representations $\tilde{\zeta}$ and ζ are *topologically equivalent* if there exists a sense-preserving homeomorphism h of S onto S such that $\tilde{\zeta} = \zeta \circ h$.

A mapping F of a disk D, $|z| < R$, is *open* if, for every open set U in D, $F(U)$ is open in the plane; F is *light* if the preimage of each image point is totally disconnected; F is *properly interior* on $\overline{D}$, $|z| \leqq R$, if F is continuous on $\overline{D}$, $F|\text{bdy } D$ is locally topological, F is sense-preserving, light and open on D. It can be shown (using results of Carathéodory, Stoilow, Whyburn) that *given a properly interior mapping F there exists an analytic function W on D that is locally topological near and on* bdy D *and there exists a sense-preserving homeomorphism H of $\overline{D}$ onto $\overline{D}$ such that $F = W \circ H$.*

A representation ζ will be called an *interior boundary* [*analytic boundary*] if ζ is locally topological and if there exists a properly interior mapping F [an analytic function W that is locally topological near and on bdy D] such that $F(Re^{it}) \equiv \zeta(t)$ [$W(Re^{it}) \equiv \zeta(t)$]. Thus, *every interior boundary is topologically equivalent to an analytic boundary.*

The problem probably first arose in the study of the Schwartz-Christoffel mapping function (Schwartz, Schlaefli, Picard) and, in this context, was formulated essentially as follows.

Let $Z_0, Z_1, \cdots, Z_{n-1}$ be a sequence of n-distinct complex numbers which are in general position. By connecting these points consecutively from Z_k to Z_{k+1}, mod n, a closed oriented polygon is formed. Let $\alpha_k\pi$ be the angle from $Z_k - Z_{k-1}$ to $Z_{k+1} - Z_k$ with $-1 < \alpha_k < 1$. Then for any set of n real number and any complex number $A \neq 0$ the function