DIFFERENTIAL POLYNOMIALS AND DECAY AT INFINITY

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Let T be an element of some space $\mathcal{H}$ of distributions on R^n. For which partial differential operators $P(D)$ with constant coefficients, may we claim that

(P) *if the support of $P(D)T$ is compact, so is the support of T itself?*

To get a reasonable answer to such a question, we must clearly impose, on the elements of $\mathcal{H}$, conditions of decay at infinity. We give here a classification of all differential polynomials, with respect to property (P), when $\mathcal{H}$ is the space of distributions rapidly decreasing at infinity (see below).

Observe that (P) holds true for any differential polynomial when $\mathcal{H}$ is the space of tempered distributions u whose Fourier transforms $\hat{u}$ can be continued to C^n as entire functions. For, to say that $P(D)u$ has compact support is the same as to say that $P(y)\hat{u}(y)$ is an entire function of exponential type. But then, $\hat{u}(y)$, being already entire, must be of exponential type (Malgrange [1, Chapter II]) and therefore $u(x)$ must have compact support.

We denote by z the variable in the complex space C^n: $z=(z_1, \cdots, z_n)$; the real space R^n will be canonically imbedded in C^n: a point z of C^n belongs to R^n if all the components z_j are reals; in this case, it is called real and denoted by $x=(x_1, \cdots, x_n)$ or $y=(y_1, \cdots, y_n)$. We consider a polynomial $P(z)$ on C^n, to which we associate the differential polynomial $P(D)$ on R^n obtained by substituting $\partial/\partial x_j$ for $2i\pi z_j$ $(1 \leqq j \leqq n)$.

Let us factorize $P(z)$ in irreducible factors:

$$P(z) = P_1(z) \cdots P_r(z).$$

For a polynomial $Q(z)$ we consider the following property:

(A) *The variety of zeros of $Q(z)$ in C^n intersects R^n.*

DEFINITION 1. *We shall say that $P(z)$ is of type* I *if all its irreducible factors have property* (A), *of type* II *if at least one of its irreducible factors does not have property* (A), *of type* III *if none of them has property* (A).

We shall make use, now, of the spaces $\mathcal{D}$, $\mathcal{S}$, $\mathcal{E}'$; for their definitions and properties, we refer to Schwartz [1]. We shall also need the space O'_C: it is the space of distributions rapidly decreasing at infinity, or, in other words, the space of tempered distributions whose Fourier transforms are C^∞ functions having, as well as all their derivatives,