A CLASS OF GEOMETRIC LATTICES

K. ROGERS AND E. G. STRAUS

Communicated by E. F. Beckenbach, December 26, 1959

1. **Introduction.** By an n-dimensional lattice Λ we mean, as usual, an additive subgroup with n linearly independent generators of the vectors in Euclidean n-space, R^n. If we denote by Z^n the lattice of vectors with integral components, then Λ is the image of Z^n under a nonsingular linear transformation:

$$\Lambda = \{Au \mid u \in Z^n\}, \qquad \det A \neq 0.$$

The matrices mapping Z^n onto Λ constitute a coset AU of the subgroup of all integral unimodular matrices, and so $\det \Lambda = |\det A|$ is well-defined. It is convenient to use the same name Λ for the point-lattice of all points P such that OP is in Λ.

Minkowski [2] showed that every lattice of determinant one contains a point other than the origin 0 in the cube

$$\{(x_1, \cdots, x_n) \mid |x_i| \leqq 1, i = 1, \cdots, n\},$$

and that the same holds if any $n-1$ of the signs are replaced by strict inequality. Those unimodular lattices, such as Z^n, which have only the origin in common with the open cube shall be called *critical*, as shall the corresponding matrices. Minkowski conjectured, and Hajos [1] proved in 1938, that a critical lattice must contain one of the points $(\delta_{i1}, \cdots, \delta_{in})$, $i=1, \cdots, n$. If A is critical then so is any matrix obtained from it by permuting rows and post-multiplication by integral unimodular matrices: such matrices will be called *equivalent* to A. An induction argument shows that Hajos' theorem is the same as the assertion:

A is critical if and only if it is equivalent to a matrix with ones on the diagonal and all zeros above.

Siegel [3] tried to prove Minkowski's conjecture by showing that, if A is critical, then each point other than 0 of the lattice corresponding to A has at least one coordinate in Z^*, the set of nonzero integers. If we consider the set of matrices A defined by the property

$$\text{(P)} \qquad u \in Z^n, \qquad u \neq 0 \Rightarrow Au \text{ has a component in } Z^*,$$

then Hajos' theorem would follow from Siegel's result, if it were true that every A with property (P) has an integral row. For in that case we could prove by induction on n that A is equivalent to a triangular matrix with zeros above the diagonal and positive integers on the