MORSE INEQUALITIES FOR A DYNAMICAL SYSTEM

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1. Introduction. We consider dynamical systems (X, M), where X is a C^∞ vector field on a C^∞ closed manifold M satisfying the following conditions.

(1) There are a finite number of singular points of X, say $\beta_1, \cdots, \beta_k$, each of simple type. This means that at each β_i, the matrix of first partial derivatives of X in local coordinates has eigenvalues with real part nonzero.

(2) There are a finite number of closed orbits (i.e., integral curves) of X, say $\beta_{k+1}, \cdots, \beta_m$, each of simple type. This means that no characteristic exponent (see, e.g., [2]) of β_i, $i > k$, has absolute value 1.

(3) The limit points of all the orbits of X as $t \to \pm \infty$ lie on the β_i. In other words, denote by ϕ_t the 1-parameter group of transformations generated by X (as we do throughout this paper). Let

$$\alpha(y) = \lim_{t \to -\infty} \text{it set } \phi_t(y), \qquad \omega(y) = \lim_{t \to \infty} \text{it set } \phi_t(y), \qquad y \in M.$$

Then for each y, $\alpha(y)$ and $\omega(y)$ are contained in the union of the β_i.

(4) The stable and unstable manifolds of the β_i (see §2 for the definition) have normal intersection with each other. More precisely for each i let W_i be the unstable manifold and W_i^* the stable manifold of β_i and for $x \in W_i$ (or W_i^*) let W_{ix} (or W_{ix}^*) be the tangent space of W_i (or W_i^*) at x. Then for each i, j if $x \in W_i \cap W_j^*$,

$$\dim W_i + \dim W_j^* - n = \dim (W_{ix} \cap W_{jx}^*).$$

See [5] for example for more details.

(5) If β_i is a closed orbit there is no $y \in M$ with $\alpha(y) = \omega(y) = \beta_i$.

First we remark that systems satisfying (1)–(5) may be very important because of the following possibilities.

(A) It seems at least plausible that systems satisfying (1)–(5) form an open dense set in the space (with the C^1 topology) of all vector fields on M.

(B) It seems likely that conditions (1)–(5) are necessary and sufficient for X to be structurally stable in the sense of Andronov and Pontrjagin [1]. See also [6].

(A) and (B) have been proved for the case M is a 2-disk, [3] and [9].

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