HANKEL TRANSFORMS AND VARIATION DIMINISHING KERNELS

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If $\phi(x)$ is a continuous function on $(-\infty, \infty)$ then we denote by $V[\phi]$ the number of variations of sign of $\phi(x)$ on $(-\infty, \infty)$. A measurable function $G(x)$ on $(-\infty, \infty)$ such that

$$G(x) \geqq 0, \qquad \int_{-\infty}^{\infty} G(x)dx = 1,$$

and such that

$$V[G * \phi] \leqq V[\phi]$$

for every bounded continuous ϕ will be called a variation diminishing $*$-kernel. Here

$$G * \phi \cdot (x) = \int_{-\infty}^{\infty} G(x-y)\phi(y)dy. \tag{1}$$

I. J. Schoenberg has proved that if G is a variation diminishing $*$-kernel then

$$\int_{-\infty}^{\infty} G(x)e^{-ixt}dx$$

is of the form

$$\left[e^{ct^2+ibt} \prod_k \left(1 - \frac{it}{a_k}\right) e^{it/a_k}\right]^{-1} \tag{2}$$

where the a_k's are real and $\sum_k a_k^{-2}$ is finite, b is real, and c is real and non-negative. Conversely every function of the form (2) is the Fourier transform of a variation diminishing $*$-kernel. See [8] and [9].

In the present note we will sketch an analogous theory in which certain convolutions of functions on $(0, \infty)$, associated with Hankel transforms replace the convolution (1).

Let γ be fixed, $0 \leqq \gamma$. We define

$$T(x) = 2^{\gamma-1/2}\Gamma(\gamma + 1/2)x^{1/2-\gamma}J_{\gamma-1/2}(x),$$

$$\mu(x) = x^{2\gamma+1}/2^{\gamma+1/2}\Gamma(\gamma + 3/2).$$

Let L be the set of measurable functions $f(x)$ on $(0, \infty)$ for which $\int_0^\infty |f(x)|\,d\mu(x)$ is finite. For $f \in L$ we set