

## A MULTIPLIER RULE FOR A CLASS OF TWO-DIMENSIONAL VARIATIONAL PROBLEMS

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We consider a calculus of variations problem of minimizing a function  $\alpha$  of integrals  $J^i$  whose integrands  $f^i$  depend on  $x, y$ , and functions  $w^i(x, y), z^k(x, y)$ , and their first partial derivatives, and possibly also on their mixed second partial derivatives  $\partial^2 w^i / \partial x \partial y, \partial^2 z^k / \partial x \partial y$ . The functions  $w^i$  and  $z^k$  defining admissible surfaces are restricted to be absolutely continuous in the sense of Vitali, and also in the sense of Tonelli, and moreover to have uniform bounds for the partial derivatives which actually enter. The functions  $z^k$  are required to take fixed boundary values on the region  $G_2$  of the  $xy$ -plane which is the domain of integration. The functions  $w^i$ , on the other hand, are restricted only by assigned initial values on two lines  $x = \bar{x}$  and  $y = \bar{y}$  which lie in  $G_2$  or on its boundary. For convenience we may suppose  $\bar{x} = \bar{y} = 0$ . Admissible surfaces are also required to give prescribed values to certain functions  $\beta^m$  of the integrals  $J^i$ , and to satisfy almost everywhere on  $G_2$  a set of partial differential equations

$$g^p(x, y, w, w_x, w_y, w_{xy}, z, z_x, z_y, z_{xy}) = 0, \quad (p = 1, \dots, p_0).$$

The functions  $f^i$  and  $g^p$  are supposed to be measurable functions of  $(x, y)$ , and to have first partial derivatives with respect to their remaining arguments which are uniformly continuous in those arguments and uniformly bounded in a suitable neighborhood of the minimizing surface.

A fundamental assumption for the proofs is that the matrix of partial derivatives of the functions  $g^p$  with respect to the arguments  $w_{xy}^j$  has at almost every point  $(x, y)$  of  $G_2$  a minor determinant of order  $p_0$  whose absolute value is greater than a fixed positive number  $\mu$ . This assumption excludes the cases treated by Gross [5], Coral [3] and others. However, the multiplier rule below does apply to the minimizing surfaces for the problems considered by Ciliberto [1] and [2], provided the partial derivatives involved are bounded. We also restrict the boundary  $G_2^*$  of  $G_2$  as follows: the portion of  $G_2^*$  lying interior to the first quadrant is defined by a continuous decreasing function  $y = \eta(x)$  on  $0 < x < a$ , and also by a continuous decreasing func-

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