ON SOLUTIONS OF RIEMANN'S FUNCTIONAL EQUATION

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1. Let $\{\lambda_n\}$, $\{\mu_n\}$ $(n \geqq 1)$ be two given sequences of positive numbers increasing to infinity, and let $\delta > 0$. We call the triplet $\{\delta, \lambda_n, \mu_n\}$ a *label*. If s is a complex variable, $s = \sigma + i\tau$, we speak of a solution of Riemann's functional equation

$$(1.1) \qquad \pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\phi(s) = \pi^{-(\delta-s)/2}\Gamma\left\{\frac{1}{2}(\delta - s)\right\}\psi(\delta - s),$$

pertaining to the label $\{\delta, \lambda_n, \mu_n\}$, if there exist two Dirichlet series $\phi(s) = \sum a_n\lambda_n^{-s}$, $\psi(s) = \sum b_n\mu_n^{-s}$ (a_n and b_n complex) which do not vanish identically, and which admit finite abscissae of absolute convergence, and a function $\chi(s)$ which is holomorphic and uniform in a domain $|s| > R$, such that $\lim_{|\tau|\to\infty}\chi(\sigma+i\tau) = 0$ uniformly in every segment $\sigma_1 \leqq \sigma \leqq \sigma_2$, and such that, for some pair of real numbers α, β, we have

$$\chi(s) = \begin{cases} \pi^{-s/2}\Gamma\left(\dfrac{1}{2}s\right)\phi(s), & \text{for } \sigma > \alpha, \\ \pi^{-(\delta-s)/2}\Gamma\left\{\dfrac{1}{2}(\delta - s)\right\}\psi(\delta - s), & \text{for } \sigma < \beta. \end{cases}$$

In three papers published recently, Bochner and Chandrasekharan [2], Chandrasekharan and Mandelbrojt [3], and Kahane and Mandelbrojt [4], have studied the problem of finding an upper bound for the number of linearly independent solutions of equation (1.1). Their results enable one to establish in certain cases a unique solution, and in certain others to deduce that the sequences $\{\lambda_n\}$, $\{\mu_n\}$ are periodic. In this note, which is a sequel to [3], we shall consider certain simple conditions which would ensure that $\delta = 1$. Let

$$D^{\mu} = \limsup\,(n/\mu_n), \qquad h_{\mu} = \liminf\,(\mu_{n+1} - \mu_n).$$

We prove the following results.

THEOREM 1. *If $h_\lambda \cdot h_\mu = 1$, δ is an odd integer, and equation* (1.1) *has a solution, then $\lambda_{n+1} - \lambda_n = h_\lambda$, and $\mu_{n+1} - \mu_n = h_\mu$, for every $n \geqq 1$. In particular, if $h_\lambda = h_\mu = 1$, δ is an odd integer, and equation* (1.1) *has a solution, then $\lambda_{n+1} - \lambda_n = 1$, and $\mu_{n+1} - \mu_n = 1$ for every $n \geqq 1$.*