THE VARIATION IN INDEX OF A QUADRATIC FUNCTION DEPENDING ON A PARAMETER

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A classical theorem states that the index for $\lambda = \mu$ of the quadratic function defined by $Q(x; \lambda) = x^T(A - \lambda I)x$ is equal to the number of characteristic roots which are less than μ. In this statement, A is a symmetric matrix of constants, x is a column vector and ^{T} denotes transpose. The function Q has the following properties: (i) it is linear in λ for each x; (ii) it has negative derivative with respect to λ for each non-null x; (iii) it is positive definite for $-\lambda$ sufficiently large. This paper is an outline of a method of relaxing all three of these restrictions to the point that the dependence of Q on λ is merely smooth.

The definitions are made in a form which applies to a space of arbitrary dimension. The theorems are restricted to spaces of finite dimension, yielding both a model and a tool for an extension to spaces of infinite dimension.

The *state* of the quadratic function for a particular parameter value is described by indices which are directly defined. The changes in state are given in terms of indices of an auxiliary quadratic function (see Theorem 3). The state of the quadratic function is described alternately in terms of a characteristic value problem and the changes in state are described alternately in terms of an auxiliary characteristic value problem.

The formulation of this problem was prompted by work of M. Morse. See particularly the exposition of a lemma of Morse in §6 of the paper [**G**].

Suppose B is a symmetric bilinear function on a vector space X, depending differentiably on a parameter λ on an interval Λ for each $(x, y) \in X \times X$. Let $Q(x; \lambda) = B(x, x; \lambda)$. The *negative index* (*index* for short) of $Q(\cdot\,; \mu)$ is the least upper bound $h(\mu)$ of the dimension of planes on which $Q(\cdot\,; \mu)$ is negative definite. The *positive index* $m(\mu)$ is the index of $-Q(\cdot\,; \mu)$. The *characteristic plane* $N(\mu)$ is defined by

$$N(\mu) = \{x \mid B(x, y; \mu) = 0 \text{ for all } y \in X\}$$

and its dimension is the *nullity* $\nu(\mu)$. The value μ is a *characteristic root* if $\nu(\mu) > 0$.

The *auxiliary function* is defined by

$$q^\mu(x) = Q_\lambda(x; \mu) \mid x \in N(\mu).$$