EXTENSIONS OF THE LEMMA OF HAAR IN THE CALCULUS OF VARIATIONS

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This note is concerned with necessary and sufficient conditions on the coefficients A_i in order that a linear functional of the form

$$L(v) = \int_G \sum_i A_i D^i v dx \tag{1}$$

shall vanish identically on a suitable class of functions v which vanish on the boundary G^* of the connected open set G in n-dimensional x-space. Here i denotes an n-dimensional vector with nonnegative integer components i_j, and

$$D^i v = \prod_{j=1}^{n} D_{x_j}^{i_j} v,$$

where D_{x_j} denotes partial differentiation with respect to x_j. The sum in (1) is taken over all vectors i with $0 \leqq i_j \leqq m_j$, where m is a fixed vector with positive integer components.

For the domain of the functional L it is convenient to take the class of all functions v of class C^∞ and having support compact on G (i.e., compact and contained in G). Then $L(v)$ is well defined when the coefficients A_i are all locally integrable in G. Also the following notations are meaningful (with exceptional sets of measure zero) for a locally integrable function f:

$$M_{x_j h_j} f(x) = \int_0^{h_j} f(y) ds, \qquad \Delta_{x_j h_j} f(x) = f(z) - f(x),$$

where $y_j = x_j + s$, $z_j = x_j + h_j$, $y_k = z_k = x_k$ for $k \neq j$, and

$$M_h^i = \prod_{j=1}^{n} M_{x_j h_j}^{i_j}, \qquad \Delta_h^i = \prod_{j=1}^{n} \Delta_{x_j h_j}^{i_j}.$$

We understand that x is a point in G, and that h is taken so small that all the points $x + ih = (x_j + i_j h_j)$ considered lie in G. We also set

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