A NEW FORM OF THE GENERALIZED CONTINUUM HYPOTHESIS

BY HERMAN RUBIN

Communicated by Paul R. Halmos, June 2, 1959

We shall prove that the following condition is equivalent to the generalized continuum hypothesis:

(*) *For all transfinite cardinals p and q, if p covers q, then for some r, $p=2^r$.*

By p covers q, we mean that $p>q$ and for no r is $p>r>q$.

The generalized continuum hypothesis is usually stated in the form that, for any transfinite cardinal p, 2^p covers p. We shall use instead the equivalent form [**2**; **4**] as the logical product of the aleph hypothesis $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ and the axiom of choice.

If the generalized continuum hypothesis holds, then (*) follows easily. For then if p and q are transfinite and p covers q, then by the axiom of choice for some α, $q=\aleph_\alpha$ and $p=\aleph_{\alpha+1}$ and so by hypothesis $p=2^q$.

Let us now proceed to the converse. First we shall prove the aleph hypothesis. Since for all α, $\aleph_{\alpha+1}$ covers $\aleph_\alpha$, we have $\aleph_{\alpha+1}=2^r$ for some r. Since $r<2^r$, r must be $\aleph_\gamma$ for some γ. Let $\beta(\alpha)$ be the smallest such γ. We clearly have $\beta(\alpha)<\alpha+1$. However, $\beta(\alpha)$ is a strictly monotone function of α and hence is greater than or equal to α. Thus $\beta(\alpha)=\alpha$ and the aleph hypothesis is proved.

Let us now demonstrate that the axiom of choice follows from (*). We first prove from the axioms of set theory the following

LEMMA.[1] *If $2^p\leqq q+\aleph_\alpha$, where p and q are transfinite, then $p<q$ or $p<\aleph_\alpha$.*

For since $p<2^p$, $p=s+t$, where $s\leqq q$ and $t\leqq\aleph_\alpha$. Then $2^p=2^s2^t$, and by [**2**] either $2^s\leqq\aleph_\alpha$ or $2^t\leqq q$. But in the first case $s+t\leqq\aleph_\alpha$ since both s and t are, and in the second case $s+t\leqq q$ since both s and t are, and in addition t is less than or equal to an aleph. Thus we have demonstrated the lemma except for the strictness of the inequalities. That follows since [**2**; **5**] if $2^p\leqq p+r$, then $2^p\leqq r$, and $p<r$, q.e.d.

For any transfinite cardinal p, let us denote by p^* the smallest aleph [**1**] not less than or equal to p. Tarski [**3**] has shown that if p is transfinite then $p+p^*$ covers p. But since by [**2**] the mapping $p\rightarrow p^*$

[1] This lemma is due to Professor A. Tarski and is an extension of the author's original argument.