MEASURE ALGEBRAS ON ABELIAN GROUPS

WALTER RUDIN[1]

The recent developments in the general field of Fourier analysis which I wish to describe, illustrate the algebraic point of view which has established itself here as well as in most other parts of analysis, and which has been most fruitful in suggesting new and interesting problems. Typically, a question will pose itself in algebraic terms, will then be transformed into a problem which can be attacked by the methods of classical analysis, and the algebraic aspect will reappear in the solution.

To cite just one example now, one may ask for the set of all automorphisms of the group algebra of the circle group; this turns out to be equivalent to the problem of finding all permutations of the set of all integers which carry Fourier series into Fourier series; the solution of this analytical problem involves arithmetic progressions, i.e., cosets of the subgroups of the additive group of the integers, and the group theoretic nature of the problem becomes apparent (the result is described in 3.3(d) below).

This is a large subject, and I will restrict myself to those topics with which I have had close personal contact. This choice will, of course, result in the omission of much that is interesting. In particular, I will say nothing about what is perhaps the most intriguing of the open problems in Fourier analysis: the problem of spectral synthesis, or, equivalently, the problem of finding all closed ideals in $L^1(G)$.

I. **Preliminaries.** 1.1. Unless the contrary is explicitly stated, any group mentioned in the sequel will be abelian and locally compact, with addition as group operation. Associated with every such group G there is a measure m (not identically 0 or ∞), the so-called *Haar measure* of G, with the following properties [6; 7; 21; 31]: m is a non-negative completely additive set function defined for all Borel sets in G, which is *regular* (i.e., $m(E)=\sup m(K)=\inf m(V)$, where K ranges over all compact subsets of E and V ranges over all open supersets of E), and which is *translation invariant*:

An address delivered before the Cambridge meeting of the Society on August 28, 1958 by invitation of the Committee to Select Hour Speakers for Summer Meetings; received by the editors November 19, 1958.

[1] Research Fellow of the Alfred P. Sloan Foundation.