THE COHOMOLOGY OF COVERING SPACES OF H-SPACES

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In this note, the cohomology of a covering space of an H-space is computed, in terms of the cohomology of the H-space, for coefficients in a field. If the characteristic of the field is different from 2, we also calculate the ring structure.

Let X be an arcwise connected space with a continuous multiplication with unit (an H-space). We will suppose that $H^q(X;\ Z_p)$ is finite dimensional for each q (singular cohomology will be used throughout). Let $\overline{X}$ be a covering space of X, π: $\overline{X}\to X$ the covering map, G the group of deck translations of $\overline{X}$ over X, i.e. the fibre of π. Then $\overline{X}$ can be given an H-space structure so that π is a multiplicative map.

Let us consider the spectral sequence of Leray-Cartan for this covering space. We can obtain it by replacing X by a homotopically equivalent space (again denoted by X) which is a fibre space over $K(G,\ 1)$ with fibre $\overline{X}$, the inclusion of $\overline{X}$ in X being homotopic to π, and the fibre map f: $X\to K(G,\ 1)$ being multiplicative. The group G acts trivially on $H^*(\overline{X};\ Z_p)$, so we have simple coefficients in E_2 (with coefficients in Z_p).

THEOREM. *Let p be an odd prime. Then $H^*(\overline{X};\ Z_p)=A\otimes E$ as rings, where $A=\pi^*(H^*(X;\ Z_p)=H^*(X;\ Z_p)/I$, I is the ideal generated by $f^*(H^*(K(G,1);Z_p))$, E is the exterior algebra on n generators $x_1,\cdots,x_n$, where the dimension of $x_i=2p^{r_i}-1$ and $2p^{r_i}$ are the dimensions of a system of generators of the kernel of f^*. If $p=2$, then the same result holds, but only as modules.*

The proof rests on the multiplicative nature of the fibre map f. This enables us to introduce a diagonal map into the spectral sequence and obtain a spectral sequence of commutative Hopf algebras [1]. Using the fact that $E_2=H^*(\overline{X};\ Z_p)\otimes H^*(K(G,\ 1);\ Z_p)$ as Hopf algebras, one can show that if $G=Z_{p^n}$, then the kernel of f^* is a polynomial ring with one generator y in dimension $2p^r$, and d^{2p^r} is the only nontrivial differential in the spectral sequence. There is an indecomposable element $x\in H^*(\overline{X};\ Z_p)$ such that $d^{2p^r}(x)=y$, and thus x is not in the image of π^*. If $G=Z$, then all d^m's are trivial and $E_2=E_\infty$, (a result due to Serre [2]), while if $G=Z_q$, q prime to p, then $H^*(K(G,\ 1);\ Z_p)=0$ and $H^*(\overline{X};\ Z_p)$ is isomorphic to $H^*(X;\ Z_p)$. The