SOME PROBABILITY LIMIT THEOREMS

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We are concerned with the partial sums $S_0=0$, $S_n=X_1+\cdots+X_n$ of identically distributed independent random variables X_i with mean zero and finite positive variance, i.e.

$$(1) \qquad E(X_i) = 0, \qquad 0 < \sigma^2(X_i) = \sigma^2 < \infty.$$

Some of the present results describe new phenomena, whereas others are refinements of known theorems. While they deal with the limiting behavior of certain functionals of the partial sums, they appear to be of a type which cannot be reduced to the study of a functional of the Wiener process. With the exception of Theorem 6, nothing but (1) will be assumed about the common distribution of the X_i.

We define the probabilities:

$$\begin{aligned} & & c_k &= \Pr[S_k > 0], & k &\geqq 1; \\ p_0 &= 1, & p_n &= \Pr[S_1 > 0, \cdots, S_n > 0], & n &\geqq 1; \\ q_0 &= 1, & q_n &= \Pr[S_1 \leqq 0, \cdots, S_n \leqq 0], & n &\geqq 1; \end{aligned}$$

and the random variables (which by virtue of (1) exist and are finite with probability 1):

$Z=$ the first positive term in the infinite sequence $S_1, S_2, \cdots$;

$N_n=$ the number of positive terms in the finite sequence $S_1, S_2, \cdots, S_n$;

$N_A(I)=$ the number of terms S_k in the infinite sequence $S_1, S_2, \cdots$, such that $S_k\in I$ and $S_i\leqq A$ for $i=1, 2, \cdots, k$;

$N_A^*(I)=$ the number of terms S_k in the infinite sequence $S_1, S_2, \cdots$, such that $S_k\in I$ and $|S_i|\leqq A$ for $i=1, 2, \cdots, k$.

Here A is a positive number and I is a closed bounded interval. $\mu(I)$ will denote the length of I when the X_i are nonlattice random variables, and the number of integers in I when the X_i are lattice random variables such that the smallest group containing all possible values of all the partial sums S_n is the group of all integers.

THEOREM 1. *The series* $\sum_1^\infty k^{-1}(1/2-a_k)$ *converges* (*conditionally*; *probably not always absolutely*).

The constant $c=\exp \sum_1^\infty k^{-1}(1/2-a_k)$ plays an important role in

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