SOLUTION OF THE EQUATION $ze^z = a$

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Communicated by Richard Bellman, December 15, 1958

The roots of the equation $ze^z = a$ $(a \neq 0)$ play a role in the iteration of the exponential function [2; 3; 11] and in the solution and application of certain difference-differential equations [1; 9; 10; 12]. For this reason, several authors [4; 5; 7; 8; 9; 12] have found various properties of some or all of the roots. Here we "solve" the equation in the following sense. We list the roots Z_n, where n takes all integral values, and define Z_n precisely for each n. We give a rapidly convergent series for Z_n for all n such that $|n| > n_0(a)$; the first few terms provide a very good approximation to Z_n. In general, n_0 is fairly small. Finally we show how to calculate each of the remaining Z_n $(-n_0 \leqq n \leqq n_0)$ numerically by giving a variety of methods to find a first approximation to Z_n and showing how to improve this to any required degree of accuracy.

We cut the complex z-plane along the negative half of the real axis and take $|\arg z| \leqq \pi$ in the cut-plane. If we put $w = z + \log z$, we have $dw/dz = (z+1)/z$ and there is a branch-point at $z = -1$. The cuts in the w-plane are the two semi-infinite lines on which $w = u \pm \pi i$, $u \leqq -1$. It can be proved that there is a one-to-one correspondence between the points of the z-plane and those of the w-plane, excluding the cuts in each case, so that the function $z(w)$ is uniquely defined in the cut w-plane.

We write $A = |a|$, take $\log A$ real and $\log a = \log A + i\alpha$, where $-\pi < \alpha \leqq i\pi$. All the roots of our equation are given by $Z_n = z(\log a + 2n\pi i)$, where n takes all integral values. Z_n is thus precisely defined except when $\alpha = \pi$ and $\log A \leqq -1$, (i.e. when a is real and $-e^{-1} \leqq a < 0$). In this one case, $\log a$ and $\log a - 2\pi i$ lie one on each of the two cuts in the w-plane; $z(\log a)$ has two real values, one less than -1 and one between -1 and 0, while $z(\log a - 2\pi i)$ has the same two values. If $-e^{-1} < a < 0$, we define Z_{-1} and Z_0 to be these two real values, distinguishing them arbitrarily by $Z_{-1} < -1 < Z_0 < 0$. If $a = -e^{-1}$, the equation (1) has a double root at $z = -1$ and we put $Z_{-1} = Z_0 = -1$. In addition, when a is real and positive, Z_0 is real. There are no other real roots for any a.

For every nonreal root Z_n, we write $Z_n = X_n + iY_n$. It is easily proved that Y_0 lies between 0 and α, that

$$(2n-1)\pi + \alpha < Y_n < 2n\pi + \alpha \qquad (n \geqq 1)$$

and that