

SOME THEOREMS CONCERNING FUNCTION ALGEBRAS

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In what follows, C will be a compact Hausdorff space, and \mathfrak{A} will be a uniformly closed separating algebra of continuous complex-valued functions on C . That is, sums, products, and complex multiples of elements of \mathfrak{A} are in \mathfrak{A} , uniform limits of elements of \mathfrak{A} are in \mathfrak{A} , and for distinct points x and y in C there exists f in \mathfrak{A} with $f(x) \neq f(y)$. Šilov (see [1]) has shown that there exists a smallest closed subset B of C , called the Šilov boundary of \mathfrak{A} , such that for each f in \mathfrak{A} there exists x in B with $|f(x)| = \|f\|$, where $\|f\| = \max \{ |f(y)| : y \in C \}$. The following theorem generalizes this result, in case C is metric.

THEOREM 1. *Let C be a compact metric space, and let \mathfrak{A} be a uniformly closed separating algebra of continuous complex-valued functions on C . Then there exists a smallest subset M of C , called the minimal boundary of \mathfrak{A} , such that for each f in \mathfrak{A} there exists x in M with $|f(x)| = \|f\|$. The set M is equal to the set M_0 which is defined as follows:*

$$M_0 = \{x: x \in C, \exists f \text{ in } \mathfrak{A} \text{ with } |f(x)| > |f(y)| \text{ for all } y \neq x \text{ in } C\}.$$

The closure of M is the Šilov boundary of \mathfrak{A} .

The question of the topological structure of M is answered by the following theorem.

THEOREM 2. *Let ρ be the metric on C . For each positive integer n , let*

$$U_n = \{x: x \in C, \exists f \text{ in } \mathfrak{A} \text{ with } \|f\| \leq 1, \\ |f(x)| > 3/4, \text{ and } |f(y)| < 1/4 \text{ for } \rho(x, y) \geq n^{-1}\}.$$

Then $\bigcap U_n = M$.

Since it is easy to see that U_n is open, it follows that M is a G_δ .

It is known that every bounded linear functional ϕ on \mathfrak{A} can be represented by a complex-valued Borel measure μ with $\|\mu\| = \|\phi\|$ on the Šilov boundary B of \mathfrak{A} . It is conjectured that μ can be taken to be a measure on the minimal boundary of \mathfrak{A} . The following theorem constitutes an important case of this conjecture.

THEOREM 3. *Let x be a point of C which is not in the minimal boundary M of \mathfrak{A} . Let \mathfrak{A} contain the constant functions. Then there exists a non-negative Borel measure μ on $B - \{x\}$ of norm 1 such that $f(x) = \int f d\mu$ for all f in \mathfrak{A} .*

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