RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

ON EMBEDDINGS OF SPHERES

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Imbed an $n-1$ sphere in an n sphere, and the complement is divided into two components. It seems that the closure of each of the resulting components should be a topological n-cell. This statement isn't true. The classical counterexample (in dimension 3) is the Alexander Horned Sphere.[1] It was conjectured, however, that if one restricts one's attention to some class of well-behaved imbeddings, then the statement is true. For instance, in the differentiable case, the Schoenflies Problem asks an even stronger question: Given $\phi: S^{n-1} \to E^n$, a differentiable imbedding of the $(n-1)$-sphere in Euclidean space, can one extend ϕ to a differentiable imbedding of the unit ball (of which S^{n-1} is the boundary) into Euclidean space?[2]

And, in fact, proofs exist for the usual categories of nice imbeddings: differentiable and polyhedral, in dimensions 1, 2, and 3.[3] The problem, then, is to prove this statement for arbitrary dimension N. Such a proof follows under a niceness condition which includes the condition of differentiability.[4]

Outline of proof. Let χ be the set of manifolds bounded by the $n-1$ sphere obtainable as the closure of a complement of a nice imbedding of S^{n-1} in S^n. Define a commutative semi-group structure in χ. (Really, it cannot be done, but just enough of a multiplication

[1] The classical such reference is Alexander's paper in the 1924 PNAS. For other amazing examples of bad imbeddings of 2-spheres in 3-space, there is an article by Artin and Fox in Volume 49 of the Annals of Mathematics.

[2] Results of Milnor (in the 1957 Annals) show that this is impossible as stated. That is, he obtains a diffeomorphism ϕ of S^6 onto itself that cannot be extended to a diffeomorphism of the unit ball in E^7 onto itself. Actually, it can be extended to a homeomorphism of the unit ball onto itself that is a diffeomorphism except at one point.

[3] There are proofs of this due to Alexander, also in the 1924 PNAS, and more recently, Moise, in the 1952 Annals.

[4] The fact that differentiable imbeddings are 'nice' in my sense is well-known, and fairly obvious. Whether or not my conditions of niceness subsume polyhedral imbeddings is an open question.