RECURSIVE EQUIVALENCE TYPES AND COMBINATORIAL FUNCTIONS[1]

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Introduction. Recursive equivalence types and isols. The theory of recursive equivalence types (R.E.T.s; see [**1**; **2**; **3**; **4**]) is a constructive counterpart of Cantor's theory of transfinite numbers. Two sets α and β of nonnegative integers are called *recursively equivalent* if one can be mapped onto the other by a one-one partial recursive function; we write $\alpha \simeq \beta$. The equivalence classes into which the class of all sets of nonnegative integers is decomposed by this equivalence relation are called *recursive equivalence types*; the R.E.T. to which a set α belongs will be denoted by Req α. The elementary arithmetic operations on R.E.T.s are defined by

$$\text{Req}\,\alpha + \text{Req}\,\beta = \text{Req}\,(\{2n \mid n \in \alpha\} + \{2n+1 \mid n \in \beta\}),$$
$$\text{Req}\,\alpha \cdot \text{Req}\,\beta = \text{Req}\,\{2^m \cdot 3^n \mid m \in \alpha \;\&\; n \in \beta\}.$$

It is easy to establish the existence and uniqueness of sums and products so defined, and to prove the formulas $(A+B)+C=A+(B+C)$, $A+B=B+A$, $(AB)C=A(BC)$, $AB=BA$, $A(B+C)=AB+AC$, $A+0=A$, $AB=0 \leftrightarrow (A=0 \text{ or } B=0)$, where 0 is the R.E.T. of the empty set. Two finite sets are recursively equivalent if and only if they have the same number of elements; thus it is permissible to identify the R.E.T.s of finite sets with the nonnegative integers. The R.E.T.s are partially ordered by the relation $A \leqq B$ which holds when $A+C=B$ for some R.E.T. C.

Amongst R.E.T.s a special role is played by those types A for which $A \neq A+1$; these types are called *isols* and the sets they characterize, *isolated sets*. Isolated sets are the constructive analogues of sets which are finite in the sense of Dedekind; they are precisely those sets which contain no infinite *recursively* enumerable subset. The isols are a proper subcollection of the R.E.T.s, and the nonnegative integers are a proper subcollection of the isols.

Arithmetical formulas of certain forms hold automatically for isols (and sometimes for R.E.T.s generally) provided they hold for nonnegative integers. So far ([**2**; **4**]) this has only been observed for formulas involving addition, an exponentiation multiplication. The

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