SOLUTION OF THE DIRICHLET PROBLEM FOR EQUATIONS NOT NECESSARILY STRONGLY ELLIPTIC

BY MARTIN SCHECHTER

Communicated by E. Hille, August 7, 1958

Let $\mu=(\mu_1, \mu_2, \cdots, \mu_n)$ be a sequence of indices and set

$$|\mu| = \sum \mu_k, \quad D^\mu = \partial^{|\mu|}/(i\partial x_1)^{\mu_1}(i\partial x_2)^{\mu_2} \cdots (i\partial x_n)^{\mu_n},$$

$$\xi^\mu = \xi_1^{\mu_1}\xi_2^{\mu_2} \cdots \xi_n^{\mu_n}$$

where $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ is any n-dimensional vector. The linear partial differential operator

$$A = \sum_{|\mu| \leqq m} a_\mu(x) D^\mu$$

with complex coefficients a_μ is elliptic at a point x if

$$P(x, \xi) \equiv \sum_{|\mu|=m} a_\mu(x)\xi^\mu \neq 0$$

for all real $\xi \neq 0$. It is strongly elliptic there if there is a complex constant γ such that $\operatorname{Re} \gamma P(x, \xi) \neq 0$ for $\xi \neq 0$. Let G be a bounded domain in n-space and let f and u_0 be smooth complex functions defined in G. The Dirichlet problem (A, f, u_0) is to find a complex function u such that $Au=f$ in G and all derivatives of $u-u_0$ of order $<m/2$ vanish on the boundary $\dot{G}$ of G. Gårding [2] and others have shown that if $\dot{G}$ and the coefficients a_μ are sufficiently smooth, a unique solution exists provided A is strongly elliptic and $a_{00\cdots 0}$ is large enough.

In this paper we extend the existence theory to include any elliptic operator for $n>2$ and to operators satisfying a root condition [5] if $n=2$. Such operators will be called properly elliptic. For $m=2$ all properly elliptic operators are strongly elliptic, but this is not the case for higher orders. For example, the operator corresponding to

$$P(x, \xi) = \xi_1^4 + \xi_2^4 - \xi_3^4 + i(\xi_1^2 + \xi_2^2)\xi_3^2$$

is not strongly elliptic.

THEOREM. *Let A be properly elliptic and denote its formal adjoint by A^*. Assume that the Dirichlet problem $(A^*, 0, 0)$ has only the solution*