LOCALLY TRIVIAL HOMOLOGY THEORIES, AND THE POINCARÉ DUALITY THEOREM

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The following is a brief account of a forthcoming memoir. A *complex with covering* is a triple $(K, \mathcal{K}, \mathrm{St})$, often written $(K, \mathcal{K})$, where (i) K is a chain complex $\{C_qK, \partial_q\}$ and $\mathcal{K}=\{K^\lambda\}$ is a collection of subcomplexes of K such that $K=\Sigma K^\lambda$, i.e. each $x\in C_qK$ is a finite sum of members of the groups C_qK^λ, $K^\lambda\in\mathcal{K}$; (ii) K is *augmented*, i.e. ∂_0 is a homomorphism of C_0K in the integers, such that $\partial_0|C_0K^\lambda$ is *onto*, for each K^λ; (iii) each K^λ lies in some sub-complex $\mathrm{St}\,K^\lambda$ of K, (e.g. $\mathrm{St}\,K^\lambda$ might be ΣK^μ over all μ with $K^\lambda\cap K^\mu$ nontrivial). $(K, \mathcal{K})$ is *free* whenever there exist sets G_q such that $G_q\cap C_qK^\lambda$ freely generates each K^λ, $(0\leqq q\leqq\infty)$.

Let $(K, \mathcal{K}, \mathrm{St}_K)$, $(J, \mathcal{J}, \mathrm{St}_J)$ be complexes with covering. A map $\pi\colon\mathcal{K}\to\mathcal{J}$ is *coherent* whenever $K^\lambda\cap K^\mu$ nontrivial implies $\pi K^\lambda\subseteq\mathrm{St}_J(\pi K^\mu)$. A *relation* $(K, \mathcal{K})\to^u(J, \mathcal{J})$ is a chain homomorphism $u_1\colon K\to J$ which preserves augmentations, together with a map $u_2\colon\mathcal{K}\to\mathcal{J}$, such that $\mathrm{Im}\,(u_1|K^\lambda)\subseteq u_2K^\lambda$ for all $K^\lambda\in\mathcal{K}$. We replace the arrow in the above relation by $\to_*$ or $\to_q$ according as $\mathrm{Im}\,(u_1\,\mathrm{St}_KK^\lambda)\subseteq u_2K^\lambda$, or $\mathrm{Im}\,(u_1|q\text{-cycles of }K^\lambda)\subseteq$boundaries of u_2K^λ, for each $K^\lambda\in\mathcal{K}$.

Now suppose there exists a diagram of relations

$$
(\dagger)\qquad
\begin{array}{ccccccccc}
(A^0,\mathcal{A}^0) & \xrightarrow{u} & (A,\mathcal{A}) & \xrightarrow[*]{v} & (B^1,\mathcal{B}^1) & \xrightarrow[0]{} & (A^1,\mathcal{A}^1) & \xrightarrow[*]{} & \cdots \\
\downarrow\sigma^0 & & & & \downarrow\tau & & \downarrow\sigma^1 & & \\
(L^0,\mathcal{L}^0) & & \xrightarrow[*]{\qquad w\qquad} & & (M^1,\mathcal{M}^1) & \xrightarrow[0]{} & (L^1,\mathcal{L}^1) & \xrightarrow[*]{} & \cdots
\end{array}
$$

$$
\begin{array}{ccccccccc}
\xrightarrow[*]{} & (B^n,\mathcal{B}^n) & \xrightarrow[n-1]{} & (A^n,\mathcal{A}^n) & \xrightarrow[*]{} & (B^{n+1},\mathcal{B}^{n+1}) & \longrightarrow & (A^{n+1},\mathcal{A}^{n+1}) \\
 & & & \downarrow\sigma^n & & & & \downarrow\sigma^{n+1} \\
\xrightarrow[*]{} & (M^1,\mathcal{M}^n) & \xrightarrow[n-1]{} & (L^n,\mathcal{L}^n) & \xrightarrow[*]{} & (M^{n+1},\mathcal{M}^{n+1}) & \xrightarrow[n]{} & (L^{n+1},\mathcal{L}^{n+1})
\end{array}
$$

(the top right arrow carrying no "n"). Suppose that $(L^0, \mathcal{L}^0)$ is free, and that there exists a *coherent* map $\gamma\colon\mathcal{L}^0\to\mathcal{A}$ such that (i) the composite maps $\mathcal{L}^0\to\mathcal{A}^q$, $\mathcal{L}^0\to\mathcal{L}^q$ are coherent; (ii) $\gamma\sigma_2^0=u_2^0$; (iii) $\tau v_2\gamma=w_2$; (iv) each square is commutative (e.g. $\tau_iv_iu_i=w_i\sigma_i^0$, $i=1, 2$). Then for each $q=0, 1, \cdots, n+1$, and abelian group G, *there exist homology and cohomology diagrams*