THE RADIUS OF UNIVALENCE OF THE ERROR FUNCTION

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We shall determine the radius of univalence of the error function

$$\operatorname{erf} z = \int_0^z e^{-t^2} dt,$$

that is, the radius of the largest open circular disk, $|z| < \rho$, in which erf z is schlicht. Some lower bounds for ρ have been obtained previously, namely:

$$\left\{\frac{1}{2}\left[(\pi^2+1)^{1/2}-1\right]\right\}^{1/2} = 1.07 \cdots, \qquad \text{[Nehari, 1]},$$

$$(\pi/2)^{1/2} = 1.25 \cdots, \qquad \text{[Rogozin, 2]},$$

the largest positive root R, of $x - \arctan x = \pi$, where $x = (4R^4-1)^{1/2}$; $R = 1.51 \cdots$, [Reade, **3**]. These bounds were obtained by different, rather general methods. Our methods are based on special properties of erf z, and were suggested by a detailed study of actual numerical values of erf z, which were computed on the IBM 704 at the National Bureau of Standards by E. Brauer and J. C. Gager.

THEOREM. *The radius of univalence of* erf z *is the minimum distance from the origin of points, not on the x-axis, for which* erf z *is real.*

Two proofs of this are given, one depending on the properties of the maps of $|z| = r$, and the other on the properties of the curves in the z-plane on which arg erf z is constant.

Our proofs have a constructive character and can be used to obtain bounds for ρ. With a small amount of hand calculation we find

$$1.5666 < \rho < 1.5858.$$

If we make use of the results of the elaborate calculations already referred to, we find that a plausible, seven decimal value of ρ is 1.5748376.

Added in proof, October 10, 1958. We have now shown that the situation is quite different if we use another normalization: the radius of univalence of $E(z) = \exp z^2 \operatorname{erf} z$ is $0.92413887 \ldots$.

REFERENCES

1. Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 545–551.