INTEGRAL REPRESENTATIONS FOR MARKOV TRANSITION PROBABILITIES

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The following results (and one or two others like them) are proved in [3]; the continuous-parameter analogues will appear in [4] and some applications will be discussed in [5]. The proofs are based on Sz.-Nagy's group of theorems [9] which show how a discrete- or continuous-parameter semigroup of contraction operators on Hilbert space can be "dilated"[1] to a similar semigroup (actually a group) of unitary operators acting on a larger Hilbert space. The Wintner-Stone theorems on the spectral representation of discrete/continuous-parameter groups of unitary operators then lead easily to integral representations for the Markov transition probabilities which appear in the matrix elements of the operator to be dilated.

I. *An irreducible Markov chain with a denumerable set of states possesses at least one "positive sub-invariant measure"* $\{m_j: j=1, 2, \cdots\}$; *i.e., there exist positive finite real numbers* m_j *such that*

$$\sum_{\alpha} m_\alpha p_{\alpha k} \leqq m_k \qquad (k = 1, 2, \cdots), \tag{1}$$

where p_{jk} *is the probability that a single transition from state* $\mathcal{E}_j$ *will lead directly to the state* $\mathcal{E}_k$. The numbers m_j need not be unique, even if we require the normalization $m_1 = 1$.

II. *If* $\{m_j: j=1, 2, \cdots\}$ *is a given positive sub-invariant measure associated with an irreducible Markov chain then the n-step transition probabilities for the latter can be uniquely represented in the form*

$$p_{jk}^{n} = (m_k/m_j)^{1/2} \oint e^{in\theta} \mu_{jk}(d\theta) \quad (n = 0, 1, 2, \cdots) \tag{2}$$

where the complex-valued Borel measures μ_{jk} *are supported by the circumference* Γ *of unit radius and are required to satisfy the Hermitean condition*

$$\mu_{kj} = \bar{\mu}_{jk} \qquad (j, k = 1, 2, \cdots). \tag{3}$$

The atoms of μ_{jk} *(of which there are none unless the chain is positive-recurrent) are located at the d dth roots of unity on* Γ *(where d is the*

[1] The terminology is due to P. R. Halmos (see [9]).