RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited.

SPLINE FUNCTIONS, CONVEX CURVES AND MECHANICAL QUADRATURE[1]

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The following lines describe some closely related results concerning the three subjects of the title. Detailed proofs will be given elsewhere.

1. **Spline functions.** Let x_+^{n-1} denote the truncated power function defined as x^{n-1} if $x \geqq 0$ and $=0$ if $x < 0$ ($n = 1, 2, \cdots$). Let ξ_ν ($\nu = 1, \cdots, k$) be a given finite sequence of increasing abscissae. By a *spline function* of degree $n-1$ we mean a function of the form

$$S_{n-1,k}(x) = P_{n-1}(x) + \sum_{\nu=1}^{k} C_\nu(x - \xi_\nu)_+^{n-1}, \tag{1}$$

where $P_{n-1}(x)$ is a polynomial of degree $\leqq n-1$. Equivalently, this function may be defined by separate polynomials of degree $\leqq n-1$ in each of the $k+1$ intervals $(-\infty, \xi_1), (\xi_1, \xi_2), \cdots, (\xi_k, \infty)$, such that the composite function has $n-2$ continuous derivatives for all real x. For $n=1$ we obtain a step-function, for $n=2$ a continuous broken-line graph and so on. The ξ_ν are called the *knots* of the spline function. The reasons for the name "spline function" are explained in [**5**, p. 67].

By adding to the spline (1) the monomial x^n we obtain a function

$$F(x) = x^n + S_{n-1,k}(x) \tag{2}$$

which we call a *monospline* of degree n and knots ξ_ν. Both splines and monosplines become polynomials if $k=0$. Much of the familiar Algebra of polynomials disappears if $k>0$, as these systems are not closed with respect to multiplication. Fortunately much of the Calculus of polynomials survives such as the relations

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