TOTALLY ORDERED COMMUTATIVE SEMIGROUPS

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Let $S(+, <)$ be a system consisting of a set S endowed with an associative binary operation $+$ and a total (=linear =simple) order relation $<$. The composition $+$ and the relation $<$ may be connected by either or both of the following conditions.

MC (*Monotone Condition*). If a and b are elements of S such that $a<b$ then $a+c\leqq b+c$ and $c+a\leqq c+b$ for all c in S.

CC (*Continuity Condition*). $(x, y)\rightarrow x+y$ is a continuous mapping of $S\times S$ into S, where S is endowed with the order topology.[2]

We shall call S an *ordered semigroup* (abbreviated "o.s.") if MC holds, and an *ordered topological semigroup* (abbreviated "o.t.s.") if CC holds. §2 below (Theorems 1–6) deals with the former, and §3 (Theorems 7–10) with the latter. An o.t.s. is an instance of a mob in the sense of A. D. Wallace [**30**].

If an o.s. S is a group with respect to $+$, then S is an ordered group, as customarily defined. In this case CC also holds. On the other hand, in each of Theorems 7–10, it turns out that MC emerges as a consequence of CC and other hypotheses. In general, however, MC and CC are independent.

An o.s. S satisfies the *strict* MC, i.e. $a<b$ implies $a+c<b+c$ and $c+a<c+b$, if and only if it is *cancellative*, i.e. $a+c=b+c$ or $c+a=c+b$ implies $a=b$.

In spite of the title, we shall not assume that S is *commutative*, i.e. $a+b=b+a$ for all a, b in S. In each of Theorems 7–10 and also Theorem 1 (Hölder 1901), commutativity will not be a hypothesis, but will be a conclusion of the theorem.

The bibliography (25 items) lists all papers known to me dealing with o.s.'s or o.t.s.'s which are not necessarily ordered groups. (Although every group is of course also a semigroup, the "theory of semigroups" does not presume to include the vastly larger theory of groups.) Items [**9**; **10**; **16**]; and [**17**] contain results on o.t.s.'s which

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[2] For order topology, see for example Garrett Birkhoff [**26**, pp. 39–41]. Numbers in brackets refer to the bibliography and general references listed at the end of the paper.