ON THE NONEXISTENCE OF ELEMENTS OF HOPF INVARIANT ONE

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With the usual definitions of homotopy-theory, we have the following theorem.

THEOREM 1. (a) *S^{n-1} is not an H-space unless $n=2$, 4, or 8.*

(b) *There is no element of Hopf invariant one in $\pi_{2n-1}(S^n)$ unless $n=2$, 4, or 8.*

For the context of this question, see [5] (especially pp. 436–438), [4, Chapter VI] and [6, §§20, 21].

This theorem results from reasonings with secondary cohomology operations. It is generally understood that a secondary operation corresponds to a relation between primary operations. One may formalize the notion of a "relation" by introducing pairs (d, z), algebraic in nature, as follows.

Let p be a prime; let A be the Steenrod algebra [2, p. 43] over Z_p. One defines the notion of a graded left module M over the graded algebra A so that $M=\sum_q M_q$ and $A_q M_r \subset M_{q+r}$. For example, let us write $H^q(X)$ for $H^q(X; Z_p)$, $H^*(X)$ for $\sum_q H^q(X; Z_p)$ and $H^+(X)$ for $\sum_{q>0} H^q(X; Z_p)$; then $H^*(X)$ and $H^+(X)$ are graded left modules over A. Let M, N be such modules; one defines the notion of an A-map $f: M \rightarrow N$ of degree r so that $f(M_q) \subset N_{q+r}$.

A pair (d, z), then, is to have the following nature. The first entry d is to be an A-map $d: C_1 \rightarrow C_0$ of degree zero. Here C_0, C_1 are to be modules in the above sense; we require, moreover, that they are locally finitely-generated and free, and that $(C_i)_q=0$ if $q<i$ $(i=0, 1)$. The second entry z is to be a homogeneous element of Ker d.

Let (d, z), then, be a pair of this sort. We call Φ a stable secondary cohomology operation associated with (d, z), if it satisfies the following axioms.

AXIOM (1). $\Phi(\epsilon)$ is defined for each A-map $\epsilon: C_0 \rightarrow H^+(X)$ of degree $m \geqq 1$ and such that $\epsilon d=0$.

Such a map ϵ is determined by its values on the elements of an A-base of C_0. It therefore corresponds to a set of elements of $H^+(X)$. In particular, if C_0 is free on one given generator c, we write $u=\epsilon c$; we may thus consider Φ as a function of one variable u, where u runs over a subset of $H^+(X)$. In this case we write $\Phi(u)$ for $\Phi(\epsilon)$.

For the next axiom, set $\deg(z)=n+1$, let $f: C_1 \rightarrow H^+(X)$ run over