A SURVEY OF THE THEORY OF SPECTRAL OPERATORS

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Dedicated to Marston Morse

1. Introduction and statement of the problem. The importance of the spectral reduction theory for bounded and unbounded selfadjoint and normal operators in Hilbert space is amply demonstrated by its diverse applications to such far reaching fields of mathematics as the theories of topological groups, almost periodic functions, harmonic analysis, and selfadjoint boundary value problems. The problems centering around the reduction theory for nonnormal operators are among the most important problems in the theory of linear operators. Notable among the many early contributions to such problems were those of I. Fredholm [20] in 1903 and G. D. Birkhoff [5] in 1908. Fredholm discussed a certain class of linear integral equations and Birkhoff, a class of linear differential boundary value problems on a finite interval. The operators discussed in the Fredholm theory are compact and have spectra which are at worst convergent sequences. The corresponding spectral resolutions need not be countably additive, or, what amounts to the same thing, the eigenvalue expansions need not be unconditionally convergent. The Fredholm theory was later given a more abstract basis, stated in operator form and free of determinant theory, by F. Riesz [32], J. Schauder [33], and T. H. Hildebrandt [22]. The deep and comprehensive work of Birkhoff on eigenvalue expansions associated with (not necessarily selfadjoint) differential operators of arbitrary order strongly suggests that, except for certain irregular cases, linear differential boundary value problems on a finite interval will have unconditionally convergent (in Hilbert space) eigenvalue expansions. That this is indeed the case is shown by the work of J. T. Schwartz[1] and H. P. Kramer [24] who have given Birkhoff's results in an abstract linear operator form. The general formulation shows that the expansion theory is valid for operators whose analytical expressions may involve integral and difference operators as well as other types of terms. The recently announced results of M. A. Neumark [28; 29; 30] on *singular* differ-

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[1] This, and other work of Schwartz referred to here will appear in *Linear operations*, Part II, by N. Dunford and J. T. Schwartz, a forthcoming volume to be published by Interscience Publishers. This volume will be referred to as L.O. II. See also [34].