A CLASS OF LATTICE ORDERED ALGEBRAS[1]

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1. Our purpose is to characterize those lattice ordered algebras which may be represented as algebras of Carathéodory functions. This work is, accordingly, a sequel to [1] where the same problem was considered for lattice ordered groups. The rings considered here are more restrictive than those of Birkhoff and Pierce in [2], where an "F-ring" is shown to be isomorphic to a subring of the direct union of totally ordered rings (but the multiplication in [2] is not necessarily that which may be expected for functions; indeed, all products may be zero. In our case, the axioms compel the algebra multiplication to conform to that of the Carathéodory functions). Brainerd [3] has considered a class of algebras which have function space representations, but his emphasis is different from ours.

2. In this section, we define a Carathéodory algebra. Let B be a relatively complemented distributive lattice. Let E be the set of forms $f=a_1\alpha_1+\cdots+a_n\alpha_n$, where $\alpha_i\in B$, a_i real, $i=1,\cdots,n$. With $f\geqq 0$ if $a_i\geqq 0$ for all i, and addition and multiplication defined by $f+g=\sum_{i=1}^{n}\sum_{j=1}^{m}(a_i+b_j)(\alpha_i\cap\beta_j)+\sum_{i=1}^{n}a_i(\alpha_i-\bigcup_{j=1}^{m}\beta_j)+\sum_{j=1}^{m}b_j(\beta_j-\bigcup_{i=1}^{n}\alpha_i)$ and $fg=\sum_{i=1}^{n}\sum_{j=1}^{m}a_ib_j(\alpha_i\cap\beta_j)$ where $f=\sum_{i=1}^{n}a_i\alpha_i$ and $g=\sum_{j=1}^{m}b_j\beta_j$, E is a lattice ordered algebra, which we call the algebra of elementary Carathéodory functions. Let $\overline{E}$ be the conditional completion of E. $\overline{E}$ is the set of bounded Carathéodory functions. In order to define the general Carathéodory function, we need the notion of carrier. In a lattice ordered group, for every $x\geqq 0$, $y\geqq 0$, we say $x\sim y$ if $x\cap z=0$ when and only when $y\cap z=0$. The equivalence classes obtained in this way are called carriers (filets by Jaffard [4]) and form a relatively complemented distributive lattice. The equivalence class to which x belongs is called the carrier of x. In $\overline{E}$, consider pairwise disjoint sequences $\{f_n\}$ whose carriers have an upper bound, and consider the formal sums $\sum f_n$. With order, addition, and multiplication defined appropriately, these formal sums constitute a lattice ordered algebra—the Carathéodory algebra C generated by B. (For details on related matters see [5; 6] and [1].)

3. Let R be an archimedean lattice ordered algebra. Then R is a lattice with positive cone P such that x, $y\in P$, $a\geqq 0$ real, implies

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