VON NEUMANN ON MEASURE AND ERGODIC THEORY

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According to a currently popular principle of classification, mathematics is the study of various "categories." A category consists of certain "objects" (e.g., groups, topological spaces) and certain "mappings" (e.g., homomorphisms, continuous functions). One possible category has measure spaces for its objects and, correspondingly, measure-preserving transformations for its mappings. The usual distinction between pure measure theory on the one hand and ergodic theory on the other hand is merely the distinction between the study of the objects and the study of the mappings of this particular category. The purpose of the following pages is to give a descriptive summary of von Neumann's contributions to this category.

Pure measure theory consists of two parts whose motivations, methods, and results are radically different in both spirit and detail; one part treats finitely additive measures and the other part insists on assuming countable additivity. A corresponding split in ergodic theory is perfectly conceivable, but it just does not happen to exist; up to now ergodic theory has been built on a countably additive foundation only. Von Neumann's most spectacular contribution to this whole circle of ideas is in ergodic theory. This is not to say that he left no mark on pure measure theory; the discovery of the relation of the problem of (finitely additive) measure to group theory, and the proof of the uniqueness of (countably additive) Haar measure in locally compact groups are mathematical accomplishments of considerable importance. There are also a couple of isolated measure-theoretic results, one pretty and startling new proof of an old theorem, and some lecture notes of expository value. Let us proceed to a slightly more technical discussion of these matters, in the following order: finitely additive measures, countably additive measures, and measure-preserving transformations.

The "problem of measure" for n-dimensional Euclidean space R^n may be stated as follows: does there exist a positive, normalized, invariant, and additive set-function on the class of all subsets of R^n? ("Positive" means non-negative, "normalized" means that the measure of the unit cube is 1, "invariant" means invariant under rigid motions, and "additive" means finitely additive.) The work of

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