THEORY OF OPERATORS
PART II. OPERATOR ALGEBRAS

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In his earliest work with operators (reported on in Part I of this article), von Neumann recognized the need for a detailed study of families of operators. Many of the subtler properties of an operator are to be found only in the internal algebraic structure of the algebra of polynomials in the operator (and its closures relative to various operator topologies) or in the action of this algebra on the underlying Hilbert space. His interest in ergodic theory, group representations, and quantum mechanics contributed significantly to von Neumann's realization that a theory of operator algebras was the next important stage in the development of this area of mathematics. The dictates of the subject itself had called for this development.

In [20], von Neumann initiated the study of the so-called "rings of operators" also called "W^*-algebras" and, most recently, "von Neumann algebras" (by Dixmier [1]). The latter term seems particularly apt, and we shall refer to them in that way.

Let us set down some notation and definitions.

DEFINITION. *A family of (bounded) operators $\mathfrak{F}$ is said to be self-adjoint when A in $\mathfrak{F}$ implies A^* is in $\mathfrak{F}$, A^* the adjoint operator to A. The 'commutant', $\mathfrak{F}'$, of $\mathfrak{F}$ is the set of those operators which commute with each operator in $\mathfrak{F}$.*

We denote by "(x, y)" the inner product of the two vectors x, y in the Hilbert space $\mathfrak{H}$ and by "$\|x\|$" the "length", $(x, x)^{1/2}$, or "norm" of the vector x. If A is an operator on $\mathfrak{H}$, the continuity of A is equivalent to its boundedness;

$$\|A\| = \sup \{\|Ax\| : \|x\| = 1\} < \infty,$$

and $\|A\|$ is called "the bound" or "norm" of A. With $d(A, B) = \|A - B\|$, d is a metric on the bounded operators, and the topology induced is called the "uniform" (also "norm" and "bound") topology. The weak operator topology is the topology on the bounded operators with the fewest open sets for which the mappings $A \to (Ax, y)$ of bounded operators into complex numbers is continuous, for each pair of vectors x, y in $\mathfrak{H}$. The strong operator topology is the topology on the bounded operators with the fewest open sets for which the mapping

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