THEORY OF OPERATORS

PART I. SINGLE OPERATORS

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The work of von Neumann on Operator Theory as distinct from the later work on *Rings of operators*, extends from 1928 to 1932 with certain later additions. It is a remarkable development in which the inadequacy of formula mathematics for quantum mechanics was established and a new abstract approach introduced. This approach required the abstract definition of Hilbert space and reformulation of the basic Hilbert theory for bounded symmetric operators. Not only was the spectral theory of Hilbert space extended to the unbounded case, for self adjoint operators, but a fascinating new range of phenomena was explored—the symmetric transformations which are not self adjoint. In addition, the foundations for the theory of rings were laid and the canonical resolution which itself was critical in further developments was obtained.

At first glance, the Gram-Schmidt process seems an easy way to replace the notion of an abstract operator by a much more elementary object, an infinite matrix. One simply takes a denumerable set dense in the domain of the operator and orthonormalizes it. The matrix then is obvious and in terms of the matrix we can simply write formulas instead of abstract relations. It certainly is surprising to learn that this matrix does not necessarily give the operator we started out with and yet this possibility must be faced as soon as one tries to specify the operator itself. The operator is determined by certain relations and closure under linearity and in the topological sense. The exact modern form of this relation is in the last paper of this series, i.e., the one on adjoint operators, but the effort to attain such a form is clear throughout. The best examples to indicate the discrepancies between the obvious formulas and the result of closure is given by a sequence of closed symmetric transformations, each an extension of the previous one and each with a dense domain. To obtain significant formulas, the abstract structure must be considered; one cannot rely on blind manipulation.

The concept of domain brings to light other critical weaknesses in the obvious formula approach. The quantum mechanics certainly posed the problem of forming the sum or the product of two opera-

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