VON NEUMANN AND LATTICE THEORY

GARRETT BIRKHOFF

1. **Introduction.** John von Neumann's brilliant mind blazed over lattice theory like a meteor, during a brief period centering around 1935–1937. With the aim of interesting him in lattices, I had called his attention, in 1933–1934, to the fact that the sublattice generated by three subspaces of Hilbert space (or any other vector space) contained 28 subspaces in general, to the analogy between dimension and measure, and to the characterization of projective geometries as irreducible, finite-dimensional, complemented modular lattices.

As soon as the relevance of lattices to linear manifolds in Hilbert space was pointed out, he began to consider how he could use lattices to classify the factors of operator-algebras. One can get some impression of the initial impact of lattice concepts on his thinking about this classification problem by reading the introduction of [**62**],[1] in which a systematic lattice-theoretic classification of the different possibilities was initiated. In particular, he saw that factors of Type II_1 gave rise to continuous-dimensional modular subspace lattices.

However, von Neumann was not content with considering lattice theory from the point of view of such applications alone. With his keen sense for axiomatics, he quickly also made a series of fundamental contributions to pure lattice theory. The primary aim of the following paragraphs is to sketch these contributions.[2]

2. **Continuous geometries.** Von Neumann's major contributions to lattice theory centered around his concepts of a "continuous geometry" and of a "regular ring." Brief announcements of his ideas appeared in the Proceedings of the National Academy of Sciences [**66**; **67**; **68**; **73**; **74**]. Full expositions were published only as mimeographed notes (*Continuous geometry*, Institute for Advanced Study Lecture Notes, Spring, 1936; *Continuous geometry*, 1936–1937, Lecture Notes, Edwards Bros., 1937). See also [**65**] and [**76**].

One can construct a continuous-dimensional projective geometry $CG(F)$ over an arbitrary division ring F, as follows. For any n, the

Received by the editors January 23, 1958.

[1] References in brackets refer to the Bibliography of *John von Neumann*, 1903–1957 which appears on page 1 of this volume.

[2] Another survey of von Neumann's work on continuous geometry, by Israel Halperin, will appear in a Hungarian journal. I have borrowed freely from this source, and am also indebted to I. Kaplansky and S. Ulam for helpful comments.