AN UNSHELLABLE TRIANGULATION OF A TETRAHEDRON

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A triangulation K of a tetrahedron T is shellable if the tetrahedra $K_1, \cdots, K_n$ of K can be so ordered that $K_i \cup K_{i+1} \cup \cdots \cup K_n$ is homeomorphic to T for $i=1, \cdots, n$. Sanderson [Proc. Amer. Math. Soc. vol. 8 (1957) p. 917] has shown that, if K is a Euclidean triangulation of a tetrahedron then there is a subdivision K' of K which is shellable; and he raises the question of the existence of a Euclidean triangulation of a tetrahedron which is unshellable. Such a triangulation will be described here.

Let T be a tetrahedron each of whose edges has length 1.

We will describe a nontrivial Euclidean triangulation K of T such that, if R is any tetrahedron of K, then the closure of $(T-R)$ is not homeomorphic to T.

I. *Construction of K*: Let X_1, X_2, X_3, and X_4 be the vertices of T.

The possible values for the letters i and j are 1, 2, 3, and 4 and addition involving i or j will be modulo 4.

For each i, let F_i denote the face of T opposite X_i, and let U_i be the midpoint of the interval X_iX_{i+2}. Observe that $U_1=U_3$ and $U_2=U_4$.

Let ϵ be the length of the shortest side of a triangle whose longest side is of length 1 and two of whose angles are 1° and 60°.

For each i, let Y_i denote the point of F_{i+1} at a distance $(3^{1/2}/2)\epsilon$ from X_i such that the angle $Y_iX_iX_{i+2}$ is 1°.

For each i, let Z_i denote the point of F_{i+2} such that the angle $Z_iX_iX_{i+3}$ is 1° and the angle $Z_iX_{i+1}X_i$ is 1°.

The fourteen vertices of our triangulation K are the points X_i, Y_i, Z_i, and U_i. It can be shown that no triangulation which has less than 14 vertices has the desired property.

The tetrahedra of our triangulation K are the tetrahedra of the forms:

(1) $X_iZ_iX_{i+1}Y_i$,
(2) $X_iZ_{i+1}X_{i+1}Y_i$,
(3) $Z_iZ_{i+1}X_{i+1}Y_i$,
(4) $Z_iZ_{i+1}X_{i+1}Y_{i+1}$,
(5) $Z_1Z_2Z_3Z_4$,
(6) $Z_iZ_{i+1}Y_iZ_{i+2}$,
(7) $X_iZ_{i+1}Y_iZ_{i+2}$,
(8) $X_iZ_{i+1}Y_{i+2}Z_{i+2}$,
(9) $X_iU_iY_{i+2}Z_{i+2}$,